Swarthmore College
Honors Exam in Geometry

May 2007

Instructions: Do as many of the following problems as you can. Justify all answers. You may quote any standard result as long as that result is not essentially what you are being asked to prove. If you do quote a standard result, make sure you clearly identify the result and verify that the hypotheses are satisfied.

In all of these problems, "smooth" means infinitely differentiable. We use the notations $\mathbb{R}^n$ for n-dimensional Euclidean space, $\mathbb{H}^2$ for the hyperbolic plane (with curvature -1), and $\mathbb{S}^2$ for the unit 2-sphere.

If you are asked to prove something about Euclidean, hyperbolic, or spherical geometry, you may base your proof either on a system of axioms or on an analytic model, but be sure to say what you are basing it on.

1. Let $P, Q \in \mathbb{H}^2$ be any two distinct points in the hyperbolic plane, and suppose $P', Q' \in \mathbb{H}^2$ are points such that the line segments PQ and $P'Q'$ have the same length. Show that there are exactly two isometries $F\colon \mathbb{H}^2 \to \mathbb{H}^2$ such that $F(P) = P'$ and $F(Q) = Q'$.

2. Let S be the surface parametrized by $X\colon U \to \mathbb{R}^3$, where

$$X(u,v) = (u, v, u^2 - v^2),$$
$$U = \{(u,v) : u^2 + v^2 < 1\}.$$

   (a) Calculate the first fundamental form of S in terms of (u, v).

   (b) Calculate the area of S.

3. Let $S \subset \mathbb{R}^3$ be a regular surface. Prove that S is complete if and only if every subset of S that is closed and bounded (in the intrinsic metric) is compact.

4. Let $S \subset \mathbb{R}^3$ be a compact, connected regular surface of positive genus. Show that there exist points $p_1, p_2, p_3 \in S$ such that $K(p_1) < 0$, $K(p_2) = 0$, and $K(p_3) > 0$, where K denotes Gaussian curvature.

5. Suppose $\varphi\colon \mathbb{S}^2 \to \mathbb{S}^2$ is a global isometry of the sphere. Show that there exists a linear map $A\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $\varphi = A|_{\mathbb{S}^2}$.

6. Let $\alpha\colon (a,b) \to \mathbb{R}^3$ be a smooth curve of the form $\alpha(t) = (0, f(t), g(t))$ with $f(t) > 0$ for all $t \in (a,b)$. Let S be the surface of revolution generated by revolving α about the z-axis. Show that the restriction of α to any closed interval $[a_0, b_0] \subset (a,b)$ is distance-minimizing.

7. Two lines in the hyperbolic plane $\mathbb{H}^2$ are said to be *asymptotically parallel* if they admit unit-speed parametrizations $\gamma_1, \gamma_2\colon \mathbb{R} \to \mathbb{H}^2$ such that $d(\gamma_1(t), \gamma_2(t))$ is bounded as $t \to +\infty$, or, equivalently, if their representations in the Poincaré disk model approach the same point on the boundary of the disk. An *ideal triangle* is a region in $\mathbb{H}^2$ whose boundary consists of three distinct lines, each pair of which are asymptotically parallel. Prove that all ideal triangles have the same finite area, and compute it.

8. Consider the following three regular surfaces in $\mathbb{R}^3$:

$$
\begin{aligned}
S_1 &= \{(x,y,0) : x, y \in \mathbb{R}\};\\
S_2 &= \{(x,y,z) : x^2 + y^2 = 1,\ 0 < z < 1\};\\
S_3 &= \{(x,y,z) : z = x^2 + y^2\}.
\end{aligned}
$$

   For each surface, answer the following questions: Is it bounded? Is it complete? Is it flat? (A surface is *flat* if each point has a neighborhood that admits an isometry with an open subset of the Euclidean plane.)