Swarthmore Honors Examination in Geometry
May 2004

Please answer 5 questions from Part I and 1 question from Part II.
Attempt more if time permits.

Part I: Differential Geometry

1. Given constants a and h, consider the cylinder $\{x^2 + y^2 = a^2\} \subset \mathbb{R}^3$ and a helix, γ, in this cylinder given by given by

$$\gamma(t) = (a\cos t, a\sin t, ht/2\pi), \qquad t \in (0, 2\pi).$$

a. Calculate the length of this helix using the standard "calculus formula"

$$\text{length}(\gamma) = \int_0^{2\pi} |v|\, dt,$$

where v denotes the velocity vector of γ.

b. Calculate the curvature of the helix. Does there exist a point of maximum or minimum curvature? If so, where?

c. Find a parameterization that covers the portion of the cylinder with $x < a$, and then find the first fundamental form (a.k.a. metric) associated with this parameterization.

d. Use the first fundamental form computed above to give an alternate calculation of the length of the helix.

2. Write a brief essay about intrinsic and extrinsic geometric quantities. Be sure to describe what is meant by an intrinsic or extrinsic measurement, and include as many examples of both as possible. Some examples you may want to consider are the
• curvature of a curve, • length of a curve, • principle curvatures of a surface,
• Gauss Curvature, • mean curvature, • covariant derivatives.
Give brief justifications for why those among your examples are intrinsic.

3. Consider the ellipsoid

$$E = \left\{ x^2 + y^2 + \frac{z^2}{4} = 1 \right\}.$$

a. Show that E is a manifold.

b. Calculate the tangent space to E at the points $(0,0,-2), (1,0,0), (1/\sqrt{2}, 0, \sqrt{2})$.

c. Compute the principal curvatures, Gaussian curvature, and mean curvature at the point $(0,0,-2)$ of E.

d. Show that E is diffeomorphic but not isometric to the sphere

$$S = \{x^2 + y^2 + z^2 = 1\}.$$

4. Consider the surface of revolution in $\mathbb{R}^3$ obtained by revolving the curve $x = f(z)$ in the (x,z)-plane about the z-axis. (Assume $f(z)$ is always positive.) One parameterization for this surface is given by

$$\mathrm{x}(\theta, z) = (f(z)\cos\theta, f(z)\sin\theta, z), \qquad 0 < \theta < 2\pi,\ \ z \in \mathbb{R}.$$

a. For this parameterization, the Christoffel symbols are given by

$$\Gamma^1_{11} = 0, \qquad \Gamma^2_{11} = \frac{-ff'}{(f')^2+1}$$

$$\Gamma^1_{12} = \frac{f'}{f}, \qquad \Gamma^2_{12} = 0$$

$$\Gamma^1_{22} = 0, \qquad \Gamma^2_{22} = \frac{f'f''}{(f')^2+1}.$$

Since time is limited, verify one of these calculations: show $\Gamma^1_{11} = 0$.

b. Consider the special case where $f(z) = \cos z + 2$. Find all geodesics in the corresponding surface of revolution.

5. Let $S \subset \mathbb{R}^3$ be a regular, compact, orientable surface which is not homeomorphic to a sphere. Prove that there are points on S where the Gaussian curvature is positive, negative, and zero. Is the analogous statement true if S is a sphere?

6. a. Give the definitions of an isometry and a local isometry.

b. Give two examples of manifolds that are locally isometric but not globally isometric.

7. Consider the cone $C \subset \mathbb{R}^3$ defined by

$$z = \sqrt{x^2 + y^2}, \qquad 0 \leq z \leq 2.$$

a. Compute the scalar geodesic curvature of a circle in C at height z.

b. The portion of C with $0 \leq z \leq 1$ is a topological disc. Show that the Gauss-Bonnet Formula does not hold for this disc.

c. Find the length of the shortest geodesic in C from the point $(0, -1, 1)$ to the point $(0, 1, 1)$.

Part II: Other Geometries

8. Let ABC be a triangle in $\mathbb{R}^2$, and let U be any point of $\mathbb{R}^2$ that is not collinear with any two of the points A, B, and C.

- Let the lines AU, BU, CU meet the lines BC, CA, AB at the points A', B', C', respectively.
- Let the lines BC and $B'C'$ meet at P, AC and $A'C'$ meet at Q, and AB and $A'B'$ meet at R.

Prove that P, Q, and R are collinear.

9. The fundamental theorem of Mobius geometry states that Mobius transformations are capable of mapping any three points to any other three points.

a. Find a Mobius transformation sending 1 to 2, 2 to 3, and 3 to -1.

b. State and prove an analogous Fundamental Theorem for Euclidean geometry.

c. Consider (P, R) where R is the set of all rotations of the plane P about the origin. Show that this is a geometry (according to the Erlanger programm). Then state and prove an analogous Fundamental Theorem for this rotational geometry.

10. Explain what is meant by a hyperbolic circle. Prove that if C denotes the circumference of a hyperbolic circle then $C = 2\pi \sinh(R)$ where R is the hyperbolic radius.