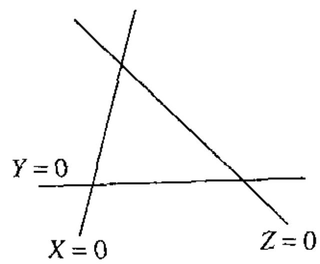
**Honors Examination in Geometry**
**May 2003**

Answer as many questions as you can. State clearly any results you rely upon. Make your responses brief but complete; explain your reasoning, and write clearly.

1. Construct the dual of the following statement, and draw figures that illustrate both the statement and the dual.

   *Statement*: A figure consists of six lines and four points in such a way that each point lies on precisely three of the lines and each line contains precisely two of the points.

2. (a) Express in terms of real homogeneous coordinates $[X, Y, Z]$ the equation of the real parabola $P : y = x^2$.
   (b) Where does $P$ meet the line at infinity $Z = 0$? What is the equation $AX + BY + CZ = 0$ of the tangent line at each such point?
   (c) Sketch $P$ in a copy of the projective plane shown at the right. Your sketch should also indicate how $P$ meets the origin $[0, 0, 1]$.
   (d) Carry out all the same steps for the real cubic curve $C$ : $y^2 = x^3$, including the shape of $C$ at $[0, 0, 1]$.




3. (a) Show there is a linear transformation

$$u = ax + by$$
$$v = cx + dy$$

   that will convert any cubic polynomial of the form

$$f(x, y) = (x + \alpha y)(x + \beta y)(x + \gamma y)$$

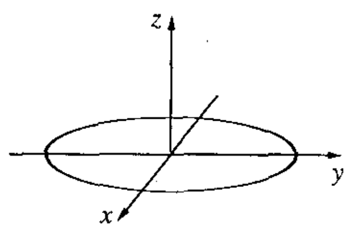
   (i.e., with three real linear factors) into the special form $F(u, v) = kuv(u + v)$.
   (b) Show that, by contrast, the invariance of the cross-ratio implies there is *no* linear transformation that will convert the quartic

$$g(x, y) = xy(x + y)(x - Py), \qquad P \neq 1,$$

   into the special form $G(u, v) = kuv(u + v)(u - v)$.

4. State Pappus's theorem and its dual.

5. The figure on the right is similar to one that appears in many mathematics texts. It shows Cartesian coordinate axes for space and the circle $x^2 + y^2 = 1$ in the $(x, y)$-plane. However, it violates a basic principle of affine geometry. What is wrong? Why is it wrong? How can you fix it?




6. If we use Cartesian coordinates in the usual way to identify the pair of real numbers $(x, y)$ with a point $P$ in the plane, then the set of points $\mathcal{C} = \{P : (x, y) | x^2 + y^2 = 1\}$ is the circle of radius 1 centered at the origin in that plane.

   Now identify the pair of real numbers $(a, b)$ with the line $L : ax + by = 1$ in the same Cartesian plane. Describe the set of lines $\mathcal{E} = \{L : (a, b) | a^2 + b^2 = 1\}$, and prove that your description is complete and accurate. Comment also on the dual nature of both the definitions and the geometric descriptions of $\mathcal{C}$ and $\mathcal{E}$.

7. Describe, in words, the image of the Gauss map of an ordinary torus (with circular core and circular cross-section). What aspect of the Gauss map indicates that the total curvature of the torus is zero?

8. (a) What are the *elliptic*, *parabolic*, and *hyperbolic* points on a smooth surface embedded in $\mathbb{R}^3$.

   (b) Consider a surface of revolution $S$ in $\mathbb{R}^3$ parametrized in the form

$$x = f(s)\cos\theta, \qquad y = f(s)\sin\theta, \qquad z = g(s),$$

   where $f > 0$ and $g$ are smooth functions and $s$ is the arc-length parameter on the generating curve $C : (x, z) = (f(s), g(s))$. Identify two different types of points on $C$ that generate circles of parabolic points on the surface $S$.

   (c) Prove your assertion in part (b).

9. The figure below shows several invariant curves for a particular Möbius transformation

$$T : \mathbb{R}^2 \cup \{\infty\} \to \mathbb{R}^2 \cup \{\infty\}.$$

The points $A$, $B$, and $C$ are on a line perpendicular to the invariant line $\mathcal{L}$, and $A' = T(A)$.

   (a) Mark $T(B)$ on the figure, and indicate clearly how you determined its location.

   (b) Similarly, mark the locations of $T(C)$ and of $T(\infty)$, the image of the point at infinity.

   (c) How many fixed points does $T$ have? Where are they?

   (d) Determine the number and location of the fixed points of the general Möbius transformation

$$T(z) = \frac{az + b}{cz + d},$$

   where $z$ is in the extended complex plane, and $a$, $b$, $c$, $d$ are complex numbers with $ad - bc \neq 0$.