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HONORS WRITTEN EXAM, DISCRETE MATHEMATICS, SPRING 2003

This is a closed book exam. No books, notes, or calculators are allowed. Write all your answers in the blue book(s). At the end of the exam, turn in the blue book(s); you may keep this exam sheet.

1. (5 points) Let n be a positive integer. What is the definition of a Latin Square of size n?

2. (10 points) Prove that the Ramsey number $r(3,3)$ equals 6.

3. (10 points) Let $A_1, \ldots, A_n$ be subsets of a set X. (We allow the possibility that some of them are equal.) A *system of distinct representatives* for these sets is an n-tuple $(x_1, \ldots, x_n)$ of elements with the properties

(a) $x_i \in A_i$ for $i = 1, \ldots, n$ (i.e., representatives);

(b) $x_i \neq x_j$ for $i \neq j$ (i.e., distinct).

Hall's (Marriage) Theorem states that the family $(A_1, \ldots, A_n)$ of finite sets has a system of distinct representatives if and only if what condition holds? Why is this called the Marriage Theorem?

4. Given a permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$ (in one-line notation) in the symmetric group S_n, we say σ has j fixed points if there are exactly j values of i, $1 \leq i \leq n$ such that $\sigma_i = i$. For example $\sigma = 326154$ in S_6 has 2 fixed points. For $1 \leq j \leq n$, let $T(n,j)$ denote the number of $\sigma \in S_n$ with j fixed points.

(a) (10 points) Using inclusion-exclusion, derive a formula for $T(n,j)$ as an alternating sum involving factorials and binomial coefficients.

(b) (5 points) Let $P(n,j)$ denote the probability that a random permutation in S_n has j fixed points, i.e., $P(n,j) = T(n,j)/n!$. Find the limit

$$\lim_{n \to \infty} P(n,j).$$

5. (10 points) For a graph $G = (V, E)$ and a positive integer x, let $p_G(x)$ denote the chromatic polynomial of G, that is, the number of ways to color the vertices of G with x colors so that no two adjacent vertices have the same color. Prove that $p_G(x)$ is a polynomial in x.

6. Let $S(n,k)$ denote the Stirling Number of the second kind, which is the number of ways to partition a set of n elements into k nonempty subsets. For example, $S(4,2)=7$, since

$$\{a,b,c,d\} = \{a,b,c\}\cup\{d\} = \{a,b,d\}\cup\{c\} = \{a,c,d\}\cup\{b\} =$$
$$\{a\}\cup\{b,c,d\} = \{a,b\}\cup\{c,d\} = \{a,c\}\cup\{b,d\} = \{a,d\}\cup\{b,c\}.$$

(a) (10 points) Prove that for $n \geq 1$ and x a positive integer

$$\sum_{k=1}^{n} S(n,k)x(x-1)\cdots(x-k+1) = x^n, \tag{1}$$

by showing that both sides of equation (1) equal $p_G(x)$ for a certain graph G with n vertices.

(b) (5 points) Prove that if equation (1) from part (a) holds for all positive integers x then it must hold for any real number x.

7. (10 points) Find a formula for the number of ways n people can form k lines (or equivalently, the number of ways to partition a set of n elements into k nonempty subsets, where the elements within any subset are ordered). For example, if $n=3$ and the integers $1,2,3$ stand for the three people, we can form 2 lines in the following 6 ways.

$$(13,2),(31,2),(12,3),(21,3),(23,1),(32,1).$$

Prove your answer gives the correct count for all positive integers n and k.

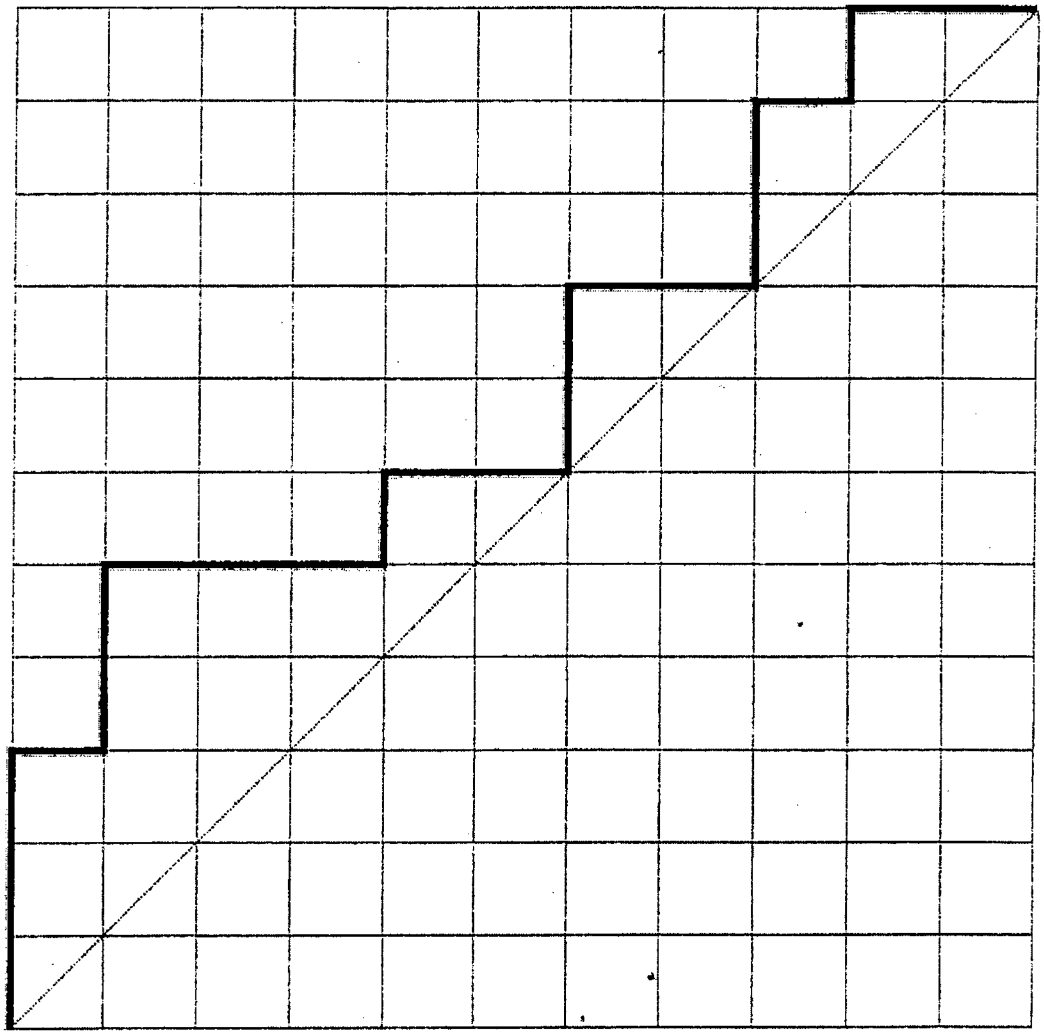
8. In this problem we consider lattice paths in the (x,y)-plane from $(0,0)$ to (n,n) consisting of North and East steps of unit length, and which never go below the line $y=x$. An example of such a path for $n=11$ is given in the Figure on the next page. For $n \geq 1$ let C_n denote the number of such paths from $(0,0)$ to (n,n), and define $C_0 = 1$.

(a) (5 points) Prove that for $n \geq 1$,

$$C_n = \sum_{k=1}^{n} C_{k-1}C_{n-k}. \tag{2}$$

(b) (10 points) Let $C(x) = \sum_{n=0}^{\infty} x^n C_n$. Assuming equation (2) from part (a) (whether you were able to prove it or not) prove that

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}, \quad |x| < \frac{1}{4}.$$



(c) (10 points) Show the result from part (b) implies

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

9. (20 points) State and prove either the Chinese remainder theorem or Fermat's "little" theorem. (You can substitute the statement and proof of any other important theorem from Math 37, but only for 15 points maximum, not 20 points.)

10. (5 points) For m a positive integer, let $\phi(m)$ denote Euler's totient function, the number of positive integers less than or equal to m that are relatively prime to m. For which positive integers m is $2^{\phi(m)} - 1$ congruent to 0 mod m?

11. (10 points) Expand $\sqrt{5}$ as an infinite simple continued fraction.

12. Let $d(n)$ denote the number of positive integers which divide n, and $[x]$ the greatest integer less than or equal to x.
(a) (5 points) Show that for any positive integer N,

$$\sum_{k=1}^{N} d(k) = \sum_{k=1}^{N} \left[\frac{N}{k}\right].$$

(b) (10 points) Show that for any positive integer N,

$$|\sum_{k=1}^{N} d(k) - N \log_e N| < 2N.$$