B

HONORS WRITTEN EXAM, DISCRETE MATHEMATICS, SPRING 2003

This is a closed book exam. No books, notes, or calculators are allowed. Write all your answers in the blue book(s). At the end of the exam, turn in the blue book(s); you may keep this exam sheet.

1. (10 points) Prove that the Ramsey number $r(3,3)$ equals 6.

2. Given a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ (in one-line notation) in the symmetric group S_n, we say σ has j fixed points if there are exactly j values of i, $1 \leq i \leq n$ such that $\sigma_i = i$. For example $\sigma = 326154$ in S_6 has 2 fixed points. For $1 \leq j \leq n$, let $T(n,j)$ denote the number of $\sigma \in S_n$ with j fixed points.

(a) (10 points) Using inclusion-exclusion, derive a formula for $T(n,j)$ as an alternating sum involving factorials and binomial coefficients.

(b) (5 points) Let $P(n,j)$ denote the probability that a random permutation in S_n has j fixed points, i.e., $P(n,j) = T(n,j)/n!$. Find the limit

$$\lim_{n\to\infty} P(n,j).$$

3. (10 points) For a graph $G = (V,E)$ and a positive integer x, let $p_G(x)$ denote the chromatic polynomial of G, that is, the number of ways to color the vertices of G with x colors so that no two adjacent vertices have the same color. Prove that $p_G(x)$ is a polynomial in x.

4. (5 points) Let $A_1, \ldots, A_n$ be subsets of a set X. (We allow the possibility that some of them are equal.) A *system of distinct representatives* for these sets is an n-tuple $(x_1, \ldots, x_n)$ of elements with the properties

(a) $x_i \in A_i$ for $i = 1, \ldots, n$ (i.e., representatives);

(b) $x_i \neq x_j$ for $i \neq j$ (i.e., distinct).

Hall's (Marriage) Theorem states that the family $(A_1, \ldots, A_n)$ of finite sets has a system of distinct representatives if and only if what condition holds?

5. (10 points) Find a formula for the number of ways n people can form k lines (or equivalently, the number of ways to partition a set

of n elements into k nonempty subsets, where the elements within any subset are ordered). For example, if $n = 3$ and the integers $1, 2, 3$ stand for the three people, we can form 2 lines in the following 6 ways.

$$(13, 2), (31, 2), (12, 3), (21, 3), (23, 1), (32, 1).$$

Prove your formula gives the correct answer for all positive integers n and k.

6. In this problem we consider lattice paths in the (x, y)-plane from $(0, 0)$ to (n, n) consisting of North and East steps of unit length, and which never go below the line $y = x$. An example of such a path for $n = 11$ is given in the Figure on the next page. For $n \geq 1$ let C_n denote the number of such paths from $(0, 0)$ to (n, n), and define $C_0 = 1$.

(a) (5 points) Prove that for $n \geq 1$,

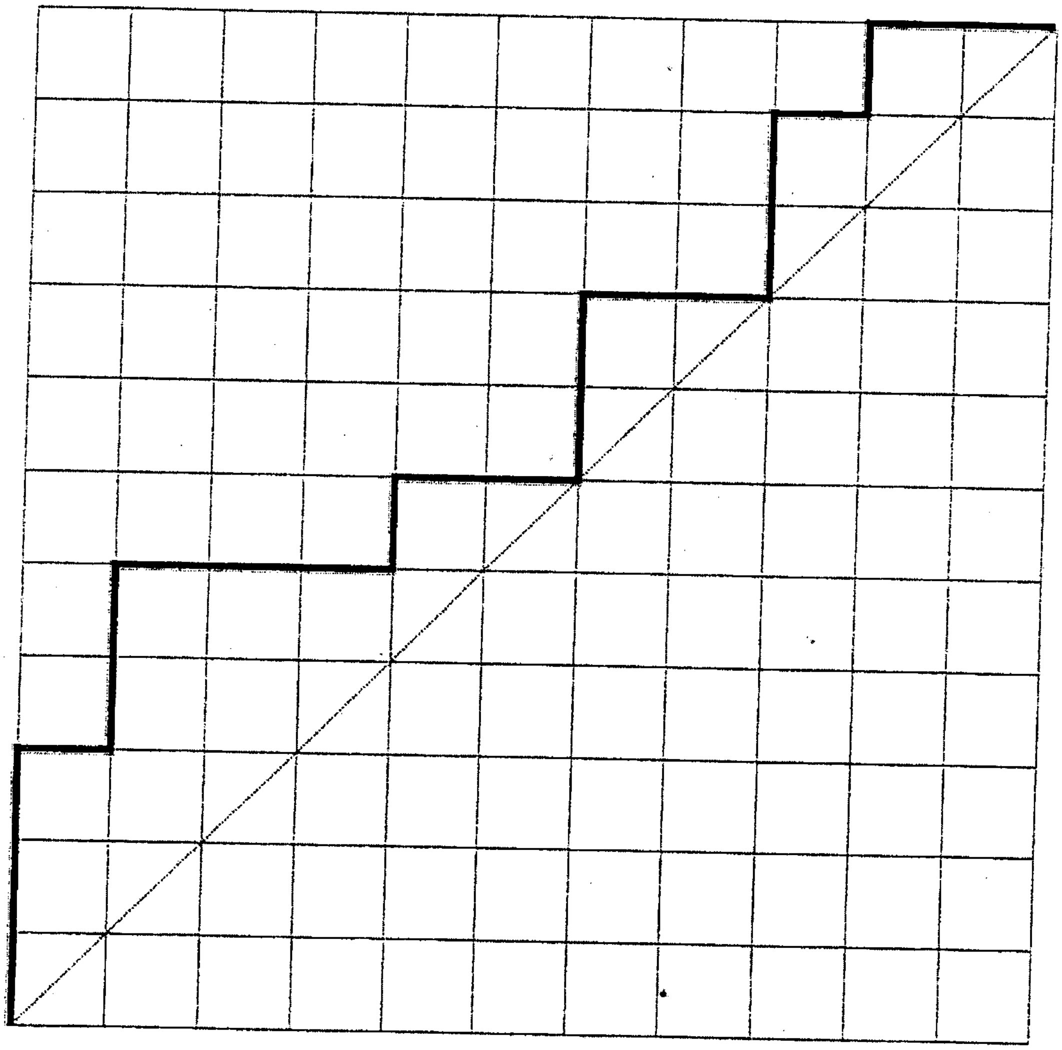
$$C_n = \sum_{k=1}^{n} C_{k-1} C_{n-k}. \tag{1}$$

(b) (10 points) Let $C(x) = \sum_{n=0}^{\infty} x^n C_n$. Assuming equation (1) from part (a) (whether you were able to prove it or not) prove that

$$C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}, \quad |x| < \frac{1}{4}.$$

(c) (10 points) Show the result from part (b) implies

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$



The "Theory of Computation Section" of this exam consists of six questions. Several of the questions contain sub-questions. The complete set contains a range of questions including some easy, some hard, some obvious and some subtle. It is very unlikely that you will have enough time to present solutions to each; I suggest that you tackle a subset that demonstrates the depth and breadth of your knowledge. The choice is meant to be yours. You should spend approximately half your time on this section of the exam.

In general, short answers that deal directly with the matter at hand will be appreciated more than lengthy ones. You may use any result from the text that you need, though you should indicate when you do so. This includes both results about specific languages and applications of more general theorems. In general, do not feel obligated to prove any result that was already proved in the text.

Good luck, stay calm, and find a way to show your best work!

Theory of Computation

1. [A few language examples]

   For each of the following three scenarios, find languages A and B such that:

   a) A and B are context-free, and $A \cap B$ is also.
   b) A and B are context-free, but $A \cap B$ is not.
   c) A and B are context-free, but not regular, while $A \cap B$ is regular.

   Note that the question asks you to identify six languages total (plus three intersections) but that duplicates are permissible.

2. [Language categorizations]

   For each of the following languages, categorize it as precisely as possible (regular, context-free, but not regular, etc.). Assume $w \in (a \cup b)^*$ unless otherwise specified.

   a) $L = \{ ww^R \mid w \in a^* \}$
   b) $L = \{ ww^R \}$
   c) $L = \{ ww^R ww^R \}$
   d) L = { w | the number of a's in w is larger than the number of b's }
   e) L = { valid C programs that take no input, but do print the word "Kelemen" }
   f) L = { C programs (compileable or not!) that some drunken student might type; note that the student may be very drunk! }

3. [Turing machines]

   Build a Turing Machine that performs the monus operation. Monus is like subtraction (aka minus), but if the difference is less than zero, then the result is defined to be zero. For example, 7 monus 4 equals 3, but 2 monus 5 equals zero. Feel free to use the standard submachines described in your text.

4. [NP-Complete (or not)]

In each off the problems below, you are given a directed graph G = (V, E) and an integer K. For each problem, determine whether or not the problem is NP-complete. If it is, prove it so; if not, describe a polynomial time algorithm for the problem. (In the latter case, a precise run-time analysis is not required; just convince me that the problem is in P.)

An elementary cycle is a sequence of vertices in a graph such that (i) there exists an edge in the graph from each vertex in the sequence to its successor, (ii) the initial and final vertices are the same, and (iii) no vertex appears as the destination of an edge (in the path) more than once.

a) Does the graph have an elementary cycle of length >= K?
b) Does the graph have an elementary cycle of length <= K?
c) Does the graph have an elementary cycle of length = |V|/2?

5. [Decidability] A useless state in a Turing machine is one that is never entered on any input string. Consider the problem of testing whether or not a Turing machine M has any useless states. Formulate this problem as a language. Then show that the language is undecidable.

6. [???] A PDA is essentially an NFA with a stack attached. Consider a new class of machine, the PTA (for Push Through Automaton). A PTA is similar to a PDA, but it has a queue attached instead of a stack. Like a PDA, it is assumed to be non-deterministic, and like a PDA, a word is accepted if it starts in the start state, uses all of the input, ends in a final state, and has an empty structure (in this case a queue instead of a stack) when execution is complete.

Formally describe a PTA, including a definition of a machine itself, a computation, and a language associated with a given machine. Include an example or two. Then address the key question: "What class of languages does a PTA accept?" Describe how this class relates to those included in the traditional complexity hierarchy.