Honors Examination in Discrete Mathematics

Swarthmore College

May, 2001

This examination consists of two parts: Combinatorics and Theory of Computation. You should spend about 90 minutes on each part. In each part, answer question #1 and at least three of the remaining questions. (Please avoid any question, if there is one, that closely duplicates your answer to either question #1.)

Answer more questions if you have time, but careful answers to fewer problems will be preferable to careless answers to more. Also, if you should find yourself at an impasse in a problem (maybe unable to fill a gap), it is far better to admit the gap and go on than it is to fudge. Doing the latter only gives the impression that you yourself have not recognized the gap.

Combinatorics

1. Choose for discussion one important theorem you have learned. Give the background, explaining why the theorem is interesting and important. Then state and prove the theorem. If it makes sense to do so, you may introduce the topic with a problem that the theorem can solve; then state and prove the theorem; then use the theorem to solve the problem.

2. Let k, n be integers ≥ 1. In the set of all k^n n–letter words on alphabet $\{1, \ldots, k\}$, find, for each $1 \leq \ell \leq k$, the number of words that contain exactly ℓ distinct letters.

3. Suppose you choose n points on a circle and draw all of the $\binom{n}{2}$ chords that connect these points. If no three of the chords are concurrent, how many points of intersection will the chords make? (Do not count the intersections on the circle at the original points; for example, if $n = 3$, the number of intersections is *zero*.)

4. For $n \geq 1$, let b_n denote the number of partitions of an n–element set. For example, $b_2 = 2$, since a two–element set admits only the indiscrete partition $\{\{a, b\}\}$ and the discrete partition $\{\{a\}, \{b\}\}$.

a. Show that if we put $b_0 = 1$, then for all $n \geq 0$,

$$b_{n+1} = \sum_{k=0}^{n} \binom{n}{k} b_{n-k}. \qquad (1)$$

b. Let

$$B(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n.$$

Translate recurrence (1) to a differential equation that the generating function B must satisfy, and solve this equation.

c. Use the closed form you obtained in part **b** to show that

$$b_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}.$$

5. Find a formula for the number of 7–bead necklaces, where each bead will be one of k possible colors. (Two necklaces count as the same necklace if there is either a rotation or a flip of the first one which will make it identical to the second one.)

6. Show that every 3–coloring of the edges of the complete graph on 17 vertices must produce a monochromatic triangle.

7. Show that there are exactly $2^{\binom{n-1}{2}}$ simple graphs on n labeled vertices in which the degree of every vertex is even. (*Hint*: Establish a bijection between these graphs and the set of all graphs on $(n-1)$ labeled vertices.)

8. Let D be a directed network with source s and sink t. Suppose that each arc (i, j) of D has a capacity c_{ij} that is a nonnegative integer. Show that D possesses a maximum (s, t) flow for which the flow x_{ij} on each arc (i, j) is also a nonnegative integer.

Throughout, $\Sigma = \{a, b\}$.

1. Choose for discussion one important theorem you have learned. Give the background, explaining why the theorem is interesting and important. Then state and prove the theorem. If it makes sense to do so, you may introduce the topic with a problem that the theorem can solve; then state and prove the theorem; then use the theorem to solve the problem.

2. For $A, B \subseteq \Sigma^*$, let us define

$$\text{splice}(A, B) := \{xyz \in \Sigma^* : xz \in A \ \& \ y \in B\}.$$

Prove that if A and B are regular languages, then so is splice(A, B).

3. Let $L = \{a^n b^n : n = 1, 2, 3, \dots\}$. Find all of the equivalence classes generated by the equivalence relation $\approx_L$.

4. Prove or disprove: the language

$$\{x \in \Sigma^* : x \neq x^R\}$$

is context free. (Recall that x^R is the reversal of word x.)

5. Prove that the language

$$\{x \in \Sigma^* : x \text{ has more } a\text{'s than } b\text{'s }\}$$

is not regular.

6. Prove that the language $\{ww: : w \in \Sigma^*\}$ is not context free. (*Hint*: First show that in the "Pumping Theorem" for CFLs, the factorization $w = uvxyz$ can be chosen so that $|vxy| \leq \phi(G)^{|V - \Sigma|}$.)

7. Suppose one modifies the definition of Turing machine so that the tape has only finitely many squares (say N squares total), and that if the machine is ever driven off of either end of the tape, it will automatically halt. Show that the halting problem for modified Turing machines is decideable.

8. Let us say that two Turing machines M_1 and M_2 are **equivalent** iff $L(M_1) = L(M_2)$. Suppose that $f : N \to N$ is a function with the property that whenever $x =$"M" for some Turing machine M, then (1), $f(x)$ is the number a Turing machine equivalent to M; and (2), $f(x)$ is the smallest number satisfying (1). Show that f is not computable.

9. The problem EXACT COVER is the following. You are given a universe $U = \{u_1, \dots, u_n\}$ and a family $\mathcal{F} = \{S_1, \dots, S_m\}$ of subsets of U. You are asked whether or not some subset of $\mathcal{F}$ constitutes a partition of the set U.

Show that EXACT COVER is $\mathcal{NP}$-complete.