Swarthmore Honors Exam: Discrete Mathematics

Spring 2000

Instructions: This is a three-hour exam, consisting of ELEVEN problems. Please choose FIVE of the problems, and try to do them as completely as possible. If you have time remaining, do as much of the rest of the exam as you can.

General hints and advice: If you get stuck on the general solution to a problem, at least compute some small special cases, and indicate what direction your thoughts are going. For example, if a question asks you to compute $f(n)$, work it out for $n = 0, 1, 2, \ldots, 6$. If you see a pattern, make a conjecture. If a question asks you to prove a theorem about graphs, verify it for all graphs with up to 4 vertices, or for other special cases like trees, cycles, etc.

1. (a) How many different ways are there of going up n stairs, if you can take them one or two at a time? *(For example, if $n = 3$, the answer is 3.)*

   (b) Let $f(n)$ denote the number of ways of going up n stairs, if you take them one, two, or three at a time. Compute $f(1), f(2), \ldots, f(6)$, and find an expression for the generating function

$$F(t) \;=\; \sum_{n=1}^{\infty} f(n)\, t^n$$

2. Let $\phi(n)$ be defined by

$$(1+t+t^2)(1+t^3+t^6)(1+t^9+t^{18})(1+t^{27}+t^{54})\cdots = \sum_{n\geq 0} \phi(n)t^n$$

   (a) Compute $\phi(n)$ for all $n \geq 0$.

   (b) Give an interpretation of this result as a statement about partitions of n.

   (c) What family $\mathcal{F}$ of partitions of n is enumerated by the function $\psi(n)$, where $\psi(n)$ is defined by

$$(1+t+t^2)(1+t^2+t^4)(1+t^3+t^6)(1+t^4+t^8)(1+t^5+t^{10})\cdots = \sum_{n\geq 0} \psi(n)t^n \;\;?$$

   Give as many descriptions of $\mathcal{F}$ as you can.

3. Let $G = (V(G), E(G))$ be a graph with $|V(G)| = n$. Let

$$\begin{aligned} \alpha(G) &= \text{maximum size of an independent set in } G \text{ (=set of pairwise nonadjacent vertices)} \\ \tau(G) &= \text{minimum size of a vertex cover in } G \text{ (=set of vertices touching all of the edges)} \end{aligned}$$

(a) What is $\alpha(G) + \tau(G)$?

(b) Show that

$$\alpha(G)\tau(G) \leq \frac{n^2}{4}$$

(c) Suppose that G contains no triangles. Show that

$$|E(G)| \leq \alpha(G)\tau(G)$$

(d) Can you have a graph G with 10 vertices and 25 edges, such that G contains no triangles? What if G has 26 edges?

4. The *Pascal Matrix* P_n is the $(n+1) \times (n+1)$ matrix defined by

$$P_n = (p_{ij})_{0 \leq i \leq n,\ 0 \leq j \leq i}$$

where p_{ij} is the binomial coefficient $\binom{i}{j}$. *(Note that the indexing of rows and columns begins with 0, and also that $\binom{i}{j} = 0$ if $i < j$.)* For example,

$$P_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}$$

Define the row sums r_i and column sums c_j, for $0 \leq i, j \leq n$, by

$$r_i = \sum_j p_{ij}, \quad c_j = \sum_i p_{ij}$$

and the diagonal sums d_k and e_k, for $0 \leq k \leq n$, by

$$d_k = \sum_{i=k}^{n} p_{i,i-k}, \quad e_k = \sum_{i=0}^{n-k} p_{i,n-k-i}$$

Thus, in the above example,

$$(r_0, r_1, r_2, r_3) = (1, 2, 4, 8) \qquad (c_0, c_1, c_2, c_3) = (1, 2, 4, 8)$$

$$(d_0, d_1, d_2, d_3) = (4, 6, 4, 1) \qquad (e_0, e_1, e_2, e_3) = (3, 2, 1, 1)$$

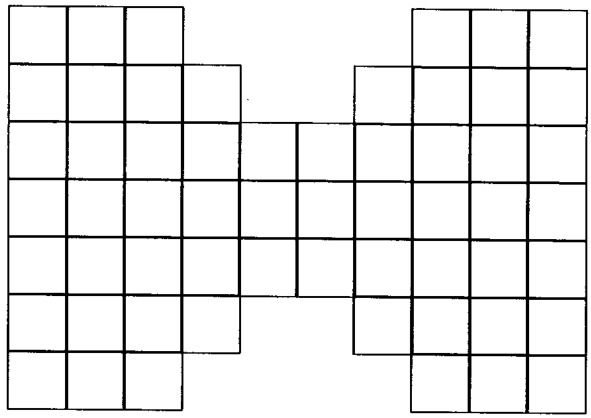
(a) Determine r_i, c_j, d_k, and e_k in general.

(b) What is P_n^{-1}, the matrix inverse of P_n?

Justify your answers.

5. Let $\mathcal{R}$ be a region in $\mathbf{R}^2$ consisting of a finite collection of of unit lattice squares (i.e., unit squares whose vertices are lattice points). *(You might want to consider $\mathcal{R}$ as a finite collection of squares on an infinite checkerboard.)* A *domino* is a pair of adjacent unit lattice squares sharing a common edge. We say that $\mathcal{R}$ can be *tiled by dominos* if there exists a covering of $\mathcal{R}$ by dominos with disjoint interiors. For example, an ordinary 8×8 checkerboard can be tiled by dominos, but it is well-known that the region obtained by removing two diagonally opposite corners cannot be tiled.

   (a) Can the following region be tiled by dominos?



   Extra copies of this diagram appear at the end of the exam, for scratch work.

   (b) Describe a general algorithm for determining whether a region $\mathcal{R}$ can be tiled by dominos. Discuss the complexity of your algorithm.

6. A *composition of n into k positive parts* is a solution $x_1, x_2, \ldots, x_k$ in positive integers to the equation

$$x_1 + x_2 + \cdots x_k = n$$

   Let $c(n,k)$ denote the number of compositions of n into k positive parts.

   (a) Give a formula in which $c(n,k)$ is expressed as a single binomial coefficient.

   (b) Show that, for each $k > 0$,

$$\sum_{n \geq 0} c(n,k) t^n = \left(\frac{t}{1-t}\right)^k$$

   Hint: you may (or may not) wish to do part (b) before part (a).

   (c) Let $CE(n)$ denote the number of compositions of n into an even number of positive parts, and let $CO(n)$ denote the number of compositions of n into an odd number of positive parts. Show that $CE(n) = CO(n)$ for all $n > 1$.

7. Suppose that G is a graph with $n \geq 3$ vertices, embedded on the surface of a sphere.

   (a) If every face of G is a triangle (i.e., every region on the sphere is bounded by exactly three edges), show the number of edges of G is exactly $3n - 6$.

   (b) Show that, if every face of G has at least three edges (i.e., there are no multiple edges), then G has at most $3n - 6$ edges.

   (c) If every face of G is bounded by at least four edges, show that the number of edges of G is at most $2n - 4$.

   (d) Is it possible to have a graph G on a sphere in which every face is bounded by at least five edges? How about six edges?

8. A *pairing* on the set $[n] = \{1, 2, \ldots, n\}$ is a partition of $[n]$ into disjoint 2-sets. For example, $\{\{1,5\},\{2,3\},\{4,6\}\}$ is a pairing on $\{1,2,3,4,5,6\}$. Let $f(n)$ denote the number of pairings on $\{1, 2, \ldots, n\}$.

   (a) Compute $f(1), f(2), f(3), \ldots, f(6)$.

   (b) Show that $f(2k) = 1 \cdot 3 \cdot 5 \cdots (2k-1)$ and $f(2k+1) = 0$, for all $k > 0$.

   (c) Show that
$$\sum_{n \geq 0} f(n) \frac{t^n}{n!} = e^{t^2/2}.$$

9. (a) Prove that
$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^m = \begin{cases} 0 & \text{if } 0 \leq m < n \\ n! & \text{if } m = n \end{cases}$$
   (Hint: you can interpret the left-hand side as an inclusion-exclusion involving functions $f : M \to N$, where $|M| = m$ and $|N| = n$.)

   (b) What happens if $m > n$?

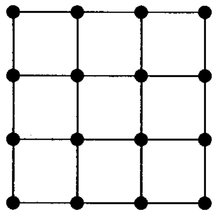
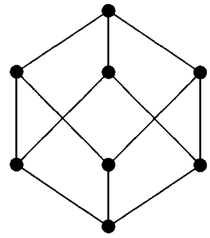
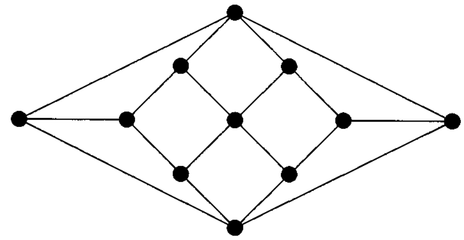
10. (a) Give an example of a graph G whose chromatic number is $\chi(G)$ is not equal to its clique number $\omega(G)$ (= size of largest clique in G).

   (b) Give at least two examples of large families $\mathcal{F}$ of graphs such that $\chi(G) = \omega(G)$ for all $G \in \mathcal{F}$.

   (c) Discuss the validity of the following statement: *If a graph G satisfies $\chi(G) = \omega(G)$, then $\chi(\bar{G}) = \omega(\bar{G})$.* Here $\bar{G}$ denotes the complement of G, i.e., the graph on the same vertex set whose edges are the non-edges of G.

   (d) Is it possible to find a graph whose clique number $\omega(G) = 2$ (i.e., it has no triangles), and $\chi(G)$ is as large as, say, 20? If not, explain why not; if so, sketch how you might construct such a graph.

11. (a) For each of the following graphs G, determine whether G has a Hamiltonian cycle.

The first two graphs are examples of *product graphs of paths*. For example, the first graph is $P_4 \times P_4$, where P_4 denotes a path with four vertices. The second is $P_2 \times P_2 \times P_2$, where P_2 denotes a path with two vertices.

(b) For which m and n does $P_m \times P_n$ have a Hamiltonian cycle?

(c) For which k does $P_2 \times P_2 \times P_2 \times \cdots \times P_2$ (k times) have a Hamiltonian cycle?

(d) Can you state any other interesting results about Hamiltonian cycles?

Extra copies of the diagram for problem # 5.)