Honors Examination in
Differential Geometry and Differential Topology

Swarthmore College
Department of Mathematics and Statistics
Tuesday May 14, 1996

Directions: The exam consists of 10 questions in three parts. Please do **2** problems from Part I, **3** problems from Part II, and **1** problem from Part III. Attempt more problems if time permits.

Part I: Choose 2 from Problems 1–3.

1. The set of 2×2 matrices, $M(2)$, is a manifold diffeomorphic to $\mathbf{R}^4$. Consider

$$SL(2) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2) \,\middle|\, \det(M) = ad - bc = 1 \right\}.$$

a. Prove that $SL(2)$ is a submanifold of $M(2)$.

b. Verify that the tangent space to $SL(2)$ at the identity matrix consists of all matrices with trace (i.e., sum of the diagonal entries) equal to 0.

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2. Which of the following linear spaces intersect transversally?

- The (x, y) plane and the z-axis in $\mathbf{R}^3$;
- $\mathbf{R}^2 \times \{0\}$ and the diagonal in $\mathbf{R}^2 \times \mathbf{R}^2$;
- $\mathbf{R}^2 \times \{0\}$ and the graph $\{(x, f(x))\}$ of $f\colon \mathbf{R}^2 \to \mathbf{R}^2$, $f(x_1, x_2) = (x_1^2, x_1 x_2)$ in $\mathbf{R}^2 \times \mathbf{R}^2$;
- The symmetric 2×2 matrices ($A^t = A$) and the skew symmetric 2×2 matrices ($A^t = -A$) in $M(2)$ (as defined in problem 1 above).

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3. If X, Z are two compact, oriented submanifolds of the oriented manifold Y, let $I(X, Z)$ denote the oriented intersection number of the inclusion map of X with Z.

a. Show that $I(X, Z) = (-1)^{(\dim X)(\dim Z)} I(Z, X)$.

b. Show that the Euler characteristic of an odd-dimensional, compact, oriented manifold is zero.

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Part II: Choose 3 from Problems 4–7.

4. Consider the parametrized curve

$$\alpha(t) = \left(2e^{-t}\cos t, 2e^{-t}\sin t\right), \qquad t \in \mathbf{R}.$$

a. Show that α is a regular differentiable curve.

b. Sketch the trace of α.

c. Show that the arclength of α for $t \in [t_0, \infty)$ is finite for any fixed t_0.

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5. Consider the paraboloid $\mathcal{P} \subset \mathbf{R}^3$ given by the equation $z = x^2 + y^2$.

a. Find a local parametrization of $\mathcal{P}$.

b. Calculate the first fundamental form of $\mathcal{P}$.

c. Describe the region of the unit sphere covered by the image of the Gauss map.

d. Calculate the Gaussian curvature, K, and the mean curvature, H, of $\mathcal{P}$ at the origin.

e. Show that no neighborhood of the origin in $\mathcal{P}$ may be isometrically mapped into a plane.

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6. Use the Gauss-Bonnet theorem to prove:

a. a compact surface of positive curvature is homeomorphic to a sphere;

b. if there exist two simple closed geodesics Γ_1, Γ_2 on a compact, connected surface S of positive curvature, then Γ_1 and Γ_2 intersect.

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7. Construct an isometry of the cylinder so that the fixed point set contains precisely two points.

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Part III: Choose 1 from Problems 8–10.

8. Let $C = C(t) = (x(t), y(t), z(t)) \subset \mathbf{R}^3$ be a smooth oriented simple closed curve parametrized by arclength with non-vanishing curvature (i.e., $\kappa(t) = |C''(t)| \neq 0$). Then if $N(t)$ denotes the unit principal normal vector,

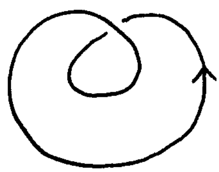
$$N(t) = C''(t)/|C''(t)|,$$

a slight displacement, $C_\epsilon(t)$, of $C(t)$ is defined by $C_\epsilon(t) = C(t) + \epsilon N(t)$. For sufficiently small ϵ, the curve C_ϵ will be simple and disjoint from C. For such an ϵ, the *self-linking number* of C is defined to be the oriented intersection number of C_ϵ and D, where D is a compact manifold with boundary, $\partial D = C$.

a. Explain why this self-linking number does not depend on ϵ for ϵ sufficiently small.

b. Calculate the self-linking number of $C(t) = (\cos(t), \sin(t), 0)$, $t \in [0, 2\pi]$.

c. Calculate the mod 2 self-linking number of the curve sketched below.



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9. **a.** Let X and Z be transversal submanifolds of Y. Prove that if $y \in X \cap Z$ then

$$T_y(X \cap Z) = T_y(X) \cap T_y(Z).$$

b. Prove that if two regular surfaces $S_1, S_2 \subset \mathbf{R}^3$ intersect transversally then $S_1 \cap S_2$ is a regular curve.

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10. Given a function $f\colon \mathbf{R}^2 \to \mathbf{R}$, the gradient vector field of f, $\operatorname{grad} f$, is defined as

$$\operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

A zero, p, of $\operatorname{grad} f$ is *non-degenerate* if $d(\operatorname{grad} f)_p\colon T_p\mathbf{R}^2 \to T_0\mathbf{R}^2$ is bijective.

a. Prove that non-degenerate zeros of $\operatorname{grad} f$ are isolated.

b. Prove that p is a non-degenerate zero of $\operatorname{grad} f$ if and only if p is a non-degenerate critical point of f.

c. Calculate the indices of all zeros of the gradient vector field of $f(x, y) = x^2 - xy$.

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