Do the best that you can to respond to the following problems. It is not required that you respond to every problem, and, in fact, it is much better to provide complete and clearly explained responses to a subset of the given problems than to provide hurried and incomplete responses to all of them. When a complete response is not possible you are encouraged to clearly explain the partial progress that you can achieve.

1. The Cauchy Condensation Test claims that if (a_n) is a nonnegative nonincreasing sequence of real numbers whose limit is 0, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} \frac{1}{2^n} a_{2^n}$ converges.
   (a) Use the Cauchy Condensation Test to determine for which $p > 0$ the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.
   (b) Is the statement still true if we no longer require (a_n) to be nonincreasing? (All other hypotheses remain.)
2. Consider the standard Cantor set in $C \subset [0,1]$. Show that $C + C = \{x + y : x, y \in C\} = [0,2]$.(Hint: If $C_1 := [0, \frac{1}{3}] \bigcup [\frac{2}{3}, 1]$, then $C_1 + C_1 = [0,2]$.)
3. Assume Young's Inequality: If a and b are nonnegative real numbers, and $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then $ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$. Prove Hölder's Inequality for sums: If $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and if $x, y \in \mathbb{R}^n$, then $|\sum_{i=1}^{n} x_n y_n| \leq (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} (\sum_{i=1}^{n} |y_i|^q)^{\frac{1}{q}}$.
4. Provide an example of a sequence of continuous functions on $[0,1]$, $(f_n(x))$, that converge pointwise to a continuous function, $f(x)$, on $[0,1]$, but such that $\lim_{n\to\infty} \int_0^1 f_n(x)dx \neq \int_0^1 f(x)dx$. What additional hypothesis beyond pointwise convergence, other than uniform convergence, would guarantee $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$?
5. Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial in the complex plane, and assume $a_n \neq 0$. Show that there are positive real constants c_1,c_2, and R such that $c_1 |z|^n \leq |p(z)| \leq c_2 |z|^n$ for all $|z| \geq R$.
6. Find all values of $(-2)^i$.
7. Suppose that f is analytic on a domain and maps that domain into a line. Show that f is a constant.

8. Prove that if u and v are harmonic conjugates, then uv is harmonic.

9. Find a nontrivial harmonic function $\phi(x, y)$ on the *wedge* domain $\{(x, y) : 0 < y < x\}$ such that $\phi = 0$ on the boundaries.

10. Show that if $f(z)$ has an essential singularity at a point z_0, then $e^{f(z)}$ also has an essential singularity at this point.

11. Show that if the Taylor Series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ converges at the point z_1, then it also converges at any z_2 such that $|z_0 - z_2| < |z_0 - z_1|$.

12. Find an explicit formula for the analytic function $f(z)$ that has the Maclaurin expansion $\sum_{k=0}^{\infty} k^2 z^k$.

13. Compute the integral $\int_{|z|=1} e^{\frac{1}{z}} \sin(\frac{1}{z}) dz$.

14. Let $p(z)$ be a polynomial and let Γ be a simple closed contour. Show that $\frac{1}{2\pi i} \int_\Gamma \frac{p'(z)}{p(z)} dz = N$, where N is the number of zeros of $p(z)$ in the interior of Γ, counted according to multiplicity.

15. Deformation Theorem: Assume that $f(z)$ is analytic in a domain D and that $z : [0, 1] \times [0, 1] \to D$ is a smooth function in both variables. Show that $h(s) = \int_{\Gamma_s} f(z) dz$ is constant where Γ_s is the contour parametrized by $z(s, t)$, $0 \le t \le 1$, for a fixed s.