Do the best that you can to respond to the following problems. It is not required that you respond to every problem, and, in fact, it is much better to provide complete and clearly explained responses to a subset of the given problems than to provide hurried and incomplete responses to all of them. When a complete response is not possible you are encouraged to clearly explain the partial progress that you can achieve.

1 Real Analysis

1. Let (x_n) be a sequence of real numbers. Show that the sequence (x_n) converges if and only if the series $\sum_{n=1}^{\infty}(x_{n+1} - x_n)$ converges.

2. Examples: Provide an example with justification, or explain why no such example exists. It is allowed to describe functions using neatly drawn graphs rather than formulas.

   (a) A continuous function $f : \mathbb{R} \to \mathbb{R}$ and a closed set $C \subseteq \mathbb{R}$ such that $f^{-1}(C)$ is open.

   (b) A sequence of bounded functions, (f_n), such that $f_n : [0, 1] \to \mathbb{R}$ and $\lim f_n(x)$ does not exist for any $x \in [0, 1]$.

   (c) A series $\sum_{n=1}^{\infty} a_n$ where the ratio test indicates convergence, but the root test does not.

3. Prove the following version of L'Hospital's Rule: Assume that f and g are continuous on $[0, 1]$, differentiable on $(0, 1)$, and $f(0) = g(0) = 0$. Show that
$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{f'(x)}{g'(x)},$$
if the latter limit exists. You may assume standard theorems of one variable calculus.

4. Consider $\mathbb{R}^N$ with the standard Euclidean norm $||x||_2$. Let $K \subset \mathbb{R}^N$ be compact and let $f : K \to K$ satisfy $||f(x) - f(y)|| < ||x - y||$ for all $x, y \in K$. Show that there is a unique point $x \in K$ such that $f(x) = x$.

5. Let l^∞ represent the set of all bounded sequences of real numbers and for $(x_n) \in l^\infty$ let $||(x_n)||_\infty := \sup\{|x_n| : n \in \mathbb{N}\}$ be a norm on l^∞. Does l^∞ possess a countable dense subset?

2 Complex Analysis

1. Find the Laurent series for $\frac{(z+1)}{z(z-4)^3}$ in $0 < |z-4| < 4$.

2. Verify the result of Picard's Theorem for $f(z) = e^{1/z}$, i.e. that, with possibly one exception, $f(z)$ assumes every complex number as a value in any neighborhood of 0.

3. Show that the projections of the points z and $-\frac{1}{\bar{z}}$ are diametrically opposite points on the Riemann sphere.

4. Prove that the function $\arg(z) Log(|z|)$ is harmonic. (Note: $Log(r)$ represents the natural logarithm of a real number r.)

5. Suppose that f is an entire function and $|f(z)| \leq c|z|^k$ for some $c > 0$ and some positive integer k. Show that f is a polynomial.

6. Show that if f is analytic on a domain D and maps D onto a subset of a line, then f is constant on D.

7. Use the Residue Theorem to compute

$$\int_0^{2\pi} \frac{d\theta}{2 - \cos\theta}$$