COMPLEX ANALYSIS HONORS EXAM 2013

This exam consists of two sections, on real and complex analysis, with four problems in each. Do as many of the problems as you can, but attempt at least two from each section. Please share your thoughts on the problem even if they do not solve it completely. In multiple-part problems, you can turn in your solution even if you can't do all the parts, and you can answer a later part without solving the earlier ones.

1. Real Analysis

(1) Let $\mathcal{K}$ be the family of all bounded sets in $\mathbb{R}$. For every $K \in \mathcal{K}$, denote by

$$\operatorname{diam} K = \sup\{|x-y| : x, y \in K\}$$

its diameter, and, for $K_1, K_2 \in \mathcal{K}$, denote by

$$u(K_1, K_2) = \inf\{|x-y| : x \in K_1, y \in K_2\}$$

their naive distance (which may well be zero). Prove that

$$d(K_1, K_2) = \begin{cases} 0, & K_1 = K_2, \\ u(K_1, K_2) + \frac{1}{2}\operatorname{diam} K_1 + \frac{1}{2}\operatorname{diam} K_2, & K_1 \neq K_2 \end{cases}$$

defines a metric on $\mathcal{K}$.

(2) Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume additionally that:

- For every $x \in \mathbb{R}$, the function $f_x : \mathbb{R} \to \mathbb{R}$ defined by $f_x(y) = f(x, y)$ is a uniformly continuous function of y, and
- For every $y \in \mathbb{R}$, the function $f^y : \mathbb{R} \to \mathbb{R}$ defined by $f^y(x) = f(x, y)$ is a uniformly continuous function of x.

Must f be uniformly continuous?

(3) Let $a < b$, let I be an open interval containing $[a, b]$, and let $f : I \to \mathbb{R}$ be a continuous function such that, for every $c \in [a, b]$, the limit

$$f'_+(c) = \lim_{x \to c+} \frac{f(x) - f(c)}{x - c}$$

exists (as a finite real number).

(a) If $f'_+(c) > 0$ for every $c \in [a, b]$, then prove that then $f(b) \geqslant f(a)$. (Hint: Use the Axiom of Completeness, also known as the Axiom of Supremum.)

(b) Prove the previous claim under the weaker assumption that $f'_+(c) \geqslant 0$ for every $c \in [a, b]$. (Hint: Construct another function that is very close to f in some sense and satisfies the condition of (a).)

(4) Let ℓ^1 be the set of all sequences $\mathbf{a} = (a_n)_{n=1}^{\infty}$ (with real terms a_n) such that

$$\sum_{n=1}^{\infty} |a_n| < \infty.$$

Then

$$d_1(\mathbf{a}, \mathbf{b}) = \sum_{n=1}^{\infty} |a_n - b_n| \quad \text{and} \quad d_2(\mathbf{a}, \mathbf{b}) = \sum_{n=1}^{\infty} \frac{|a_n - b_n|}{2^n}$$

are metrics on ℓ^1. Let

$$L = \left\{ \mathbf{a} \in \ell^1 : a_n \geqslant 0 \text{ for all } n \in \mathbb{N}, \ \sum_{n=1}^{\infty} a_n = 1 \right\}.$$

(a) Is L a closed set in the metric d_1? Is it bounded? Is it compact?
(b) Find a sequence in L that converges to $\mathbf{0} = (0, 0, \dots)$ in the metric d_2.
(c) Prove that, whenever $(\mathbf{a}_k)_{k=1}^{\infty}$ is a sequence of elements $\mathbf{a}_k = (a_{kn})_{n=1}^{\infty}$ <u>in L</u> with $\lim_{k\to\infty} d_2(\mathbf{a}_k, \mathbf{0}) = 0$ and $(x_n)_{n=1}^{\infty}$ is a sequence of real numbers with $\lim_{n\to\infty} x_n = x$, then

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} a_{kn} x_n = x.$$

2. Complex Analysis

(1) Let p be a nonzero polynomial with nonnegative coefficients, $\Omega \subset \mathbb{C}$ a bounded open region, $\bar{\Omega}$ its closure, and $f : \bar{\Omega} \to \mathbb{C}$ a continuous function analytic on Ω. Prove that the function $p(|f(z)|)$ achieves a maximum on the boundary $\partial\Omega$.

(2) Find the number of zeros of the function

$$f(z) = e^{1/(2(z-1))} + 2z^4 - z$$

inside the annulus $\{z \in \mathbb{C} : \frac{1}{2} < |z| < 1\}$.

(3) Let $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\bar{U} = \{z \in \mathbb{C} : |z| \leqslant 1\}$. Prove that the series

$$f(z) = \sum_{n=1}^{\infty} \frac{z^{3^n}}{3^n}$$

converges for all $z \in \bar{U}$, and that it defines a continuous function $f : \bar{U} \to \mathbb{C}$ which is analytic on U but does not admit an analytic continuation to any connected open region V such that $U \subset V$ and $U \neq V$.

(4) Let $T = ABC$ be a closed equilateral triangle in the complex plane. Denote by T° its interior and by ∂T its boundary (the union of closed segments AB, BC, and CA), so that $T = T^\circ \cup \partial T$.

Let Ω be an open region containing T and let F_Ω be the set of all analytic functions $f : \Omega \to \Omega$ such that $f(\{A, B, C\}) = \{A, B, C\}$ and $f(\partial T) \subseteq \partial T$. Find the cardinality of the set F_Ω and describe explicitly all functions in this set.