Swarthmore Honors Exam 2011
Complex Analysis

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Part I : Real Analysis

Instructions. Please answer **three** out of the following four questions. Show all of your work. If you refer to a significant theorem, please state the theorem with all necessary hypotheses. If you answer more than three, please clearly indicate which problems you wish to be graded.

I-1 A topological space is called *separable* if it contains a countable dense subset.

(a) Show that euclidean space $\mathbb{R}^n$ is separable.

(b) Show that every compact metric space is separable.

(c) Let ℓ_∞ be the space of bounded sequences $\mathbf{a} = \{a_j\}_{j=1}^\infty$. We make ℓ_∞ into a metric space by declaring

$$d(\mathbf{a}, \mathbf{b}) = \sup_j |a_j - b_j|$$

Show that ℓ_∞ is not separable.

I-2 Let $\{f_n\}$ be a uniformly bounded sequence of continuous functions on $[a, b]$. Let

$$F_n(x) = \int_a^x f_n(t)dt$$

(a) Show that there is a subsequence of $\{F_n\}$ that converges uniformly on $[a, b]$.

(b) Each F_n is differentiable. Show by an example that the uniform limit in part (a) need not be differentiable.

I-3 Let $\{a_n\}_{n=1}^\infty$ be a positive decreasing sequence $a_n \geq a_{n+1} \geq 0$.

(a) Show that $\sum_{n=1}^\infty a_n$ converges if and only if $\sum_{k=0}^\infty 2^k a_{2^k}$ converges.

(b) Use the result in part (a) to show that the harmonic series $\sum_{n=1}^\infty (1/n)$ diverges.

(c) Use the result in part (a) to show that the series $\sum_{n=2}^\infty (1/n(\log n)^p)$ converges for $p > 1$.

I-4 Let f be a continuous function on the closed interval $[a, b]$.

(a) Show that

$$\lim_{p\to\infty} \left(\int_a^b |f(x)|^p dx \right)^{1/p} = \max_{x\in[a,b]} |f(x)|$$

(b) Give an example of a continuous function f on (a, b) where the improper integrals

$$\int_a^b |f(x)|^p dx$$

exist (i.e. are finite) for all $1 \leq p < \infty$, but

$$\lim_{p\to\infty} \left(\int_a^b |f(x)|^p dx \right)^{1/p} = \infty$$

Part II: Complex Analysis

Instructions. Please answer **three** out of the following five questions. Show all of your work. If you refer to a significant theorem, please state the theorem with all necessary hypotheses. If you answer more than three, please clearly indicate which problems you wish to be graded.

II-1 Let $f(z)$ be an entire function. Suppose there is a constant C and a positive integer d such that $|f(z)| \leq C|z|^d$ for all z with $|z|$ sufficiently large. Show that $f(z)$ is a polynomial of degree at most d.

II-2 Evaluate the integral $\displaystyle\int_0^\infty \frac{dx}{x^5+1}$.

II-3 Let $D \subset \mathbb{C}$ be the unit disk, and

$$Q = \{z \in \mathbb{C} : |\text{Im } z| < \pi/2\}$$

(a) Find a conformal mapping $f : D \to Q$ with $f(0) = 0$, $f'(0) = 2$.

(b) Use the conformal mapping in part (a) to show that if $g : D \to Q$ is analytic with $g(0) = 0$, then $|g'(0)| \leq 1/2$.

II-4 For a complex parameter λ, $|\lambda| < 2$, consider solutions to the equation

$$z^4 - 4z + \lambda = 0 \tag{1}$$

(a) Show that there is exactly one solution $z(\lambda)$ to eqn. (1) with $|z(\lambda)| < 1$.

(b) Show that the map $\lambda \mapsto z(\lambda)$ is analytic for $|\lambda| < 2$.

(c) What is the order of vanishing of $z(\lambda)$ at $\lambda = 0$?

II-5 The Bernoulli numbers B_n are defined by the equation

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!}$$

(a) Compute B_0, B_1, and B_2.

(b) Show that

$$\pi z \cot(\pi z) = \sum_{k=0}^{\infty} (-1)^k B_{2k} \frac{(2\pi z)^{2k}}{(2k)!}$$

and that $B_{2n+1} = 0$ for $n \geq 1$.

(c) Compute the residues

$$\mathrm{Res}_{z=0}(\pi z^{-2n} \cot(\pi z))$$

for $n = 1, 2, 3, \ldots$, in terms of Bernoulli numbers.