Honors Exam in Real Analysis and Complex Analysis
Swarthmore College
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Attempt at least 3 questions from Part I and at least 3 questions from Part II. You need not prove standard results or state standard definitions to use them, but do invoke them explicitly enough that your arguments are clear (e.g. "By the definition of connected, we know that . . . ").

Part I — Real Analysis

I-1. If we equip the set E of all continuous functions $f : [0, 1] \to \mathbb{R}$ with the metric

$$d(f, g) = \max_{x \in [0,1]} |f(x) - g(x)|,$$

E becomes a complete metric space. Define

$$S = \{ f \in E \ : \ f'(x) \text{ exists for all } x \in (0, 1) \ \} .$$

Prove that S is neither open nor closed in E.

I-2. Let $\mathcal{K}$ denote the set of nonempty compact subsets of $\mathbb{R}^2$, and write d for the Euclidean metric on $\mathbb{R}^2$. Given $A \in \mathcal{K}$ and $r \geq 0$, write

$$A_r = \{ x \in \mathbb{R}^2 \ : \ d(x, y) \leq r \text{ for some } y \in A \ \}.$$

Observe that $A_0 = A$, and that $A_s \subset A_t$ whenver $s \leq t$.

a) Given $A, B \in \mathcal{K}$, define

$$\mu(A, B) = \min\{ r \geq 0 \ : \ B \subset A_r \ \}.$$

Show that this definition makes sense — that is, that the above minimum in fact exists.

b) The *Hausdorff metric on* $\mathcal{K}$ is defined as

$$\rho(A, B) = \max(\mu(A, B), \mu(B, A)).$$

Prove that ρ is in fact a metric.

c) Show, with an explicit example, that $\mu : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ is *not* a metric. (Hint: Consider the case $B \subset A$.)

I-3. Let E be the space $[0, 1]$ equipped with the *discrete metric*:

$$d(x, y) = \begin{cases} 0, & x = y; \\ 1, & x \neq y. \end{cases}$$

a) Show that any function $f : E \to \mathbb{R}$, where $\mathbb{R}$ has its usual metric, is continuous.

b) Is E connected?

c) Is E compact?

I-4. a) Prove that

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

b) Does

$$\sum_{k=1}^{\infty} \frac{k!}{k^k}$$

converge?

I-5. Given $a > 0$ and two continuous functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, let us define the function $h : [0, a] \to \mathbb{R}$ by the formula

$$h(x) = \int_{g(0)}^{g(x)} f(s)\ ds.$$

a) Show that h is continuous on $[0, a]$.

b) Show via an explicit example that h need not be differentiable on $(0, a)$. (Hint: try making f a constant function.)

c) Can you impose an additional condition on g or on f that guarantees that $h(x)$ *is* differentiable on $(0, a)$? (Simple and/or highly restrictive conditions are OK!)

Part II — Complex Analysis

II-1. Expand the function $f(z) = \frac{1}{1-z}$ about 0 and about -1 to show that

$$\sum_{k=0}^{\infty} \frac{1}{3^k} = \sum_{k=0}^{\infty} \frac{2^{k-1}}{3^k}.$$

(As part of your proof, be sure to explain why both of the above series converge.)

II-2. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is an entire nonconstant function.

a) Show that the image of f is dense — that is, given any $w \in \mathbb{C}$ and any $\epsilon > 0$, there is a $z \in \mathbb{C}$ such that $|f(z) - w| < \epsilon$.

b) Show that if $|f(z)| \to \infty$ as $|z| \to \infty$, then f is onto — that is, $f(z) = w$ has a solution for every $w \in \mathbb{C}$.

c) Give an explicit example that shows that, if the condition in part b) fails, f need not be onto.

II-3. Compute

$$\int_{-\infty}^{\infty} \frac{1}{2x^2+1}\,dx.$$

II-4. Show that $e^z + z^2 - 5$ has exactly one root in the left half plane. (Hint: consider a region whose boundary is of the form

$$[-ri, ri] \;\cup\; \{\; re^{i\theta} \;:\; \theta \in [\pi/2, 3\pi/2] \;\},$$

where $r > 0$.)

II-5. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is entire, and that the image $f(\mathbb{C})$ of f contains no nonpositive real numbers — that is,

$$f(\mathbb{C}) \cap \{a + 0i \;:\; a \leq 0\} = \emptyset.$$

Prove that f is constant. (Hint: find analytic functions g and h such that $g \circ h \circ f$ is entire with bounded image.)