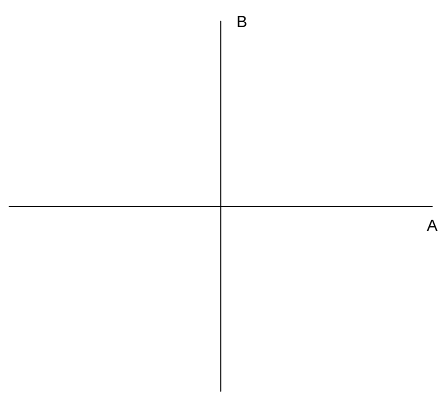
Honors Exam in Complex Analysis
Swarthmore College
Spring 2008

Answer as many questions as you can, taking care to answer questions from both parts I and II. If you need to choose between answering most questions carefully and all questions hurriedly, answer most questions carefully. You may quote any standard result as long at it is not essentially what I am asking you to prove.

Part I — Real Analysis

1. Find a metric $d(\cdot,\cdot)$ on the real line $\mathbf{R}$ that makes $\mathbf{R}$ into a bounded set: i.e. there is some $M > 0$ such that $d(x, y) \leq M$ for any $x, y \in \mathbf{R}$. Verify that d is in fact a metric.

2. Let A be the line segment $\{\ (x, 0)\ :\ -1 < x < 1\ \}$ and let B be the line segment $\{\ (0, y)\ :\ -1 < y < 1\ \}$. Suppose that $f : A \cup B \to A$ is a continuous function (view $A \cup B$ and A as subsets of the plane, with its usual metric). Prove that f cannot be one-to-one.



3. Suppose that X is a subset of a metric space E. Define the following number:

$$\alpha(X) = \inf(\delta \ : \ X \text{ can be covered by finitely many open balls in } E \text{ of radius } \delta \ \}.$$

a) Show that, if X is compact, then $\alpha(X) = 0$.

b) Show that, if X is not closed, X need not be compact even if $\alpha(X) = 0$.

(If E is complete and X is closed, then $\alpha(X) = 0$ if and only if X is compact. This is a version of the so-called *Bolzano-Weierstrass theorem*. The function $\alpha(X)$ is called a *measure of noncompactness*.)

4. If we equip the set E of all continuous functions $f : [0,1] \to \mathbf{R}$ with the metric

$$d(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|,$$

E becomes a complete metric space. Let X be the closed unit ball in this space:

$$X = \{ \text{ continuous functions } f : [0,1] \to \mathbf{R} \text{ such that } |f(x)| \leq 1 \text{ for all } x \in [0,1] \ \}.$$

Show that $\alpha(X) = 1$, where α is as in problem 3. Conclude that X, while closed and bounded, is *not* compact.

5. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a continuous function. Choose and fix some closed interval $[a,b]$. For each $k \in \mathbf{N}$, define the function

$$g_k(x) = k \int_{x-(1/k)}^{x} f(t) \ dt.$$

Show that each $g_k(x)$ is differentiable for all x, and that $g_k \to f$ uniformly on $[a,b]$ as $k \to \infty$. You may use the fundamental theorem of calculus.

(This problem shows that a continuous function can be uniformly approximated by differentiable functions on any closed interval.)

Part II — Complex Analysis

6. Find a meromorphic function f on $\mathbf{C}$ with the feature that, if γ is any smooth closed curve intersecting neither the point $a = 1 + 0i$ or $b = -1 + 0i$ in the complex plane,

$$\int_\gamma f \, dz$$

is the number of times that γ winds around a *plus* the number of times that γ winds around b (both in the positive — i.e. counterclockwise — direction).

7. Prove the Casorati-Weierstrass Theorem: Suppose f has an isolated essential singularity at p. Then the image of any neighborhood of p under f is dense in $\mathbf{C}$. Otherwise put, given $w \in \mathbf{C}$, there is a sequence $\{z_n\}$ converging to p such that $\{f(z_n)\}$ converges to w. Do not use the Picard theorems. (Hint: imagine that the image is not dense. This means that there is some w such that the function $f(z) - w$ has modulus greater than some ϵ for z near p; this in turn implies that

$$\frac{1}{f(z) - w}$$

is bounded near p.)

8. Suppose that a sequence $\{f_k\}$ of analytic functions converges normally (i.e. uniformly on compact subsets) to f on a domain $D \subset \mathbf{C}$, and that $f_k(z) \neq 0$ for every $k \in \mathbf{N}$ and every $z \in D$. Prove that f must either have no zeros or must be identically zero.

9. Let H be the upper half plane: $H = \{ \ a + bi \ : \ b > 0 \ \}$. Let $f : H \to H$ be an analytic function such that $f(i) = i$. Suppose that b and c are two real numbers greater than 1 and that $f(bi) = ci$. Show that $c \leq b$.