Swarthmore College
Honors Exam in Complex Analysis

May 2007

Instructions: Do as many of the following problems as you can. Justify all answers. You may quote any standard result as long as that result is not essentially what you are being asked to prove. If you do quote a standard result, make sure you clearly identify the result and verify that the hypotheses are satisfied.

In these problems, a function f is said to be of class C^k if f is continuous and all of its (ordinary or partial) derivatives up through order k exist and are continuous. It is said to be *entire* if it is complex analytic on all of $\mathbb{C}$.

1. Suppose K is a subset of $\mathbb{R}^n$. Prove that K is compact if and only if every continuous function $f\colon K \to \mathbb{R}$ is bounded.

2. Suppose $f\colon (0,1] \to \mathbb{R}$ is differentiable and satisfies $|f'(x)| < 1$ there. Show that the sequence $\{f(1/n)\}$ converges.

3. Suppose $f\colon [0,\infty) \to \mathbb{R}$ is of class C^2, and $f(x) \to 0$ as $x \to \infty$.
   (a) If $f'(x) \to b$ as $x \to \infty$, show that $b = 0$.
   (b) If f'' is bounded, show that $f'(x) \to 0$ as $x \to \infty$.
   (c) Give an example of such an f for which $f'(x)$ does not converge as $x \to \infty$.

4. Suppose $f\colon [0,1] \to \mathbb{R}$ is *upper semicontinuous:* This means that for every $x \in [0,1]$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that $|y-x| < \delta$ implies $f(y) < f(x) + \varepsilon$. Prove that f is bounded above and achieves its maximum value at some $x \in [0,1]$.

5. (a) Suppose $f\colon \mathbb{C} \to \mathbb{C}$ is an entire function and
$$|f(z)| \leq |z|^{3/2}$$
for all $z \in \mathbb{C}$. Show that f is identically zero.
   (b) Suppose that f is complex analytic for $0 < |z| < 1$ and $\operatorname{Re} f(z)$ is bounded there. Show that f has a removable singularity at the origin.

6. Using basic principles (e.g., considerations of uniform convergence, periodicity, and Liouville's theorem), prove that

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

7. Let $E = \{(x, y) : 0 < y < x\}$. Use conformal mapping techniques to find a harmonic function u on E with the following boundary values:

$$u(x, y) = \begin{cases} 0, & y = x \text{ or } y = 0, \text{ and } x^2 + y^2 < 1, \\ 1, & y = x \text{ or } y = 0, \text{ and } x^2 + y^2 > 1. \end{cases}$$

8. Use the residue theorem to evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{a + \cos\theta},$$

where a is a real number such that $a > 1$.