Swarthmore College
Honors Exam in Complex Analysis

May 2006

Instructions: Do as many of the following problems as you can. Justify all answers. You may quote any standard result as long as that result is not essentially what you are being asked to prove. If you do quote a standard result, make sure you clearly identify the result and verify that the hypotheses are satisfied.

In these problems, a function is said to be *entire* if it is complex analytic on all of $\mathbb{C}$. A real-valued function of one real variable is said to be C^∞ if it has continuous derivatives of all orders.

1. For what real values of p is $d_p(x, y) = |x - y|^p$ a metric on $\mathbb{R}$?

2. Suppose $\{f_n\}$ is a sequence of continuous functions on $[0, 1]$, and let $F_n(x) = \int_0^x f_n(u)\, du$ for $x \in [0, 1]$. If the functions f_n are uniformly bounded, show that some subsequence of $\{F_n\}$ converges uniformly on $[0, 1]$.

3. Let M denote the metric space whose elements are the C^∞ functions $f\colon [0, 1] \to \mathbb{R}$, with metric
$$d(f, g) = \sup\{|f(x) - g(x)| : x \in [0, 1]\}.$$
Define maps $D\colon M \to M$ and $I\colon M \to M$ by $D(f) = f'$ and $I(f) = F$, where
$$F(x) = \int_0^x f(t)\, dt.$$
Show that I is continuous but D is not.

4. Consider the function
$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{z - n}.$$
   (a) Show that f is meromorphic on $\mathbb{C}$ with poles at the positive integers.
   (b) Prove that
$$f'(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(z - n)^2}.$$

5. Find all expansions of the function
$$f(z) = \frac{1}{(z - 2)(z - 3)}$$
in powers of z, and determine where they converge.

6. Let $f(z)$ and $g(z)$ be entire functions of z for which

$$|f(z)| \leq |g(z)|.$$

Show that $f(z) = cg(z)$ for some constant c.

7. Compute

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2} dx.$$

[Hint: Find a contour that encloses just one pole.]

8. Use conformal mapping techniques to find a bounded harmonic function $u(x, y)$ defined in the semi-infinite strip

$$S = \{(x, y) \in \mathbb{R}^2 : |x| < \pi/2, \ y > 0\}$$

and satisfying the following boundary conditions:

$$\lim_{x \searrow -\pi/2} u(x, y) = 1 \qquad \text{for all } y > 0,$$
$$\lim_{x \nearrow \pi/2} u(x, y) = 1 \qquad \text{for all } y > 0,$$
$$\lim_{y \searrow 0} u(x, y) = 0 \qquad \text{for all } x \in (-\pi/2, \pi/2).$$

Does there exist an unbounded harmonic function satisfying the same conditions?