SWARTHMORE COLLEGE
Department of Mathematics and Statistics
Honors Examination

16 May 1996
1:30–4:30

Complex Analysis

DIRECTIONS: Do as many problems as you can. The term *holomorphic* is used to mean analytic in an open set. The holomorphic functions in D are denoted $\mathcal{O}(D)$; the meromorphic functions are denoted $\mathcal{M}(D)$. $\widehat{\mathbf{C}}$ is the Riemann sphere: $\mathbf{C}$ together with the point ∞.

1. Suppose that D is a domain in $\mathbf{C}$ and that $f\colon D \to \mathbf{R}$ has continuous partial derivatives.
   a) Express the condition that f defines a holomorphic function of the variable $z = x + iy$.
   b) Express the condition that f defines a holomorphic function of the variable $\zeta = x + 2iy$.

2. Suppose that D is a domain in $\mathbf{C}$ and that $f\colon D \to \mathbf{C}$ has a complex derivative at each point in D. Prove that, about each point $z_0 \in D$, f may be expressed as a power series that is absolutely convergent in a nontrivial disk about z_0. (Give enough details of the proof to show that you understand both the question and the answer.)

3. Find $\displaystyle\int_0^\infty \frac{dx}{1+x^{2n}}$ for $n \in \mathbf{N}$.

4. Suppose that $f\colon \partial\Delta \to \mathbf{C}$ is continuous. (Here, and henceforth, ∂ denotes boundary and Δ is the open unit disk.) Without getting too fancy, show (with justification) that $f(e^{i\theta}) = \sum_{n\in\mathbf{Z}} a_n e^{in\theta}$ may be used to define two functions, one of which is holomorphic in Δ and the other is holomorphic in $\widehat{\mathbf{C}} - \overline{\Delta}$. How does this differ from the formation of a Laurent series?

5. Suppose that f is holomorphic, maps the unit disk Δ into itself, and $f(0) = 0$. Can you make any assertion about the value of $f'(0)$? Can you make any assertion about the value of f' at any other points in Δ?

6. Given two complex numbers, say, ω_1 and ω_2, which are linearly independent over the reals, we may form the Weierstrass $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Omega'} \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\}$$

where $\Omega' = \{ n\omega_1 + m\omega_2 \mid (n,m) \in \mathbf{Z}^2 - \{(0,0)\} \}$. Prove that the derivative of $\wp$ may be obtained by differentiating the series term by term.

7. Suppose that f is an entire function and that there exist positive constants R, A, α so that

$$|f(z)| \leq A|z|^\alpha$$

whenever $|z| > R$. Prove that f is a polynomial and estimate its degree.

8. Suppose that $f, g \in \mathcal{O}(\mathbf{C})$ are nonconstant and that $f \circ g$ is a polynomial. Prove that both f and g are polynomials.

9. A Möbius transformation is an injective meromorphic mapping of $\widehat{\mathbf{C}}$ into itself.
   a) Prove that such a mapping must be onto.
   b) Prove that every Möbius transformation has the form

   $$\gamma : z \mapsto \frac{az+b}{cz+d}$$

   with $ad - bc \neq 0$.
   c) Determine the conditions under which two Möbius transformations commute.

10. Let z_i for $i = 1, \ldots, 4$ be four distinct points in $\widehat{\mathbf{C}}$. Determine when there exists a constant $k \in \mathbf{C}$ and a Möbius transformation γ so that $\gamma(z_1) = 0$, $\gamma(z_2) = \infty$, $\gamma(z_3) = k$ and $\gamma(z_4) = \bar{k}$. Interpret your answer geometrically.

11. Find all Laurent series expansions of the function

$$f(z) = \frac{1}{(z^2+1)(z+2)}$$

about the point $z_0 = 0$.