2025 SWARTHMORE HONORS EXAM IN COMPLEX ANALYSIS

1. Real Analysis

(1) (a) Let C be a closed subset of $\mathbb{R}$. Show that there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $C = f^{-1}(0)$.
(b) The Extreme Value Theorem says the following: let C be a compact subset of $\mathbb{R}$, and let $f : C \to \mathbb{R}$ be a continuous function. Then f attains a global minimum on C.
Prove the following converse statement: Let S be a subset of $\mathbb{R}$ with the property that every continuous function $g : S \to \mathbb{R}$ attains a global minimum on S. Then S must be a compact set.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function.
(a) Let $x \in \mathbb{R}$ with $f'(x) > 0$. Prove that there exists an $\varepsilon > 0$ such that f is strictly increasing on the interval $(x - \varepsilon, x + \varepsilon)$.
(b) Suppose f attains a local minimum at $x \in \mathbb{R}$. Prove that $f'(x) = 0$.

(3) (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a non-constant polynomial of odd degree. Prove that f has a zero in $\mathbb{R}$.
(b) Let $g : \mathbb{R} \to \mathbb{R}$ be a polynomial of even degree with positive leading coefficient. Prove that g has a global minimum on $\mathbb{R}$.

(4) Let S be a set, and let $(a_s)_{s \in S}$ be a collection of non-negative real numbers indexed by the set S. Then we define the sum

$$\sum_{s \in S} a_s$$

as the supremum

$$\sup_{F \subseteq S} \sum_{s \in F} a_s$$

where the supremum is taken over all finite subsets F of S. If the supremum does not exist, we formally define the sum to have value $+\infty$.
(a) When the set S is the natural numbers $\mathbb{N}$, prove that this definition agrees with the usual definition of the sum:

$$\sum_{i=1}^{\infty} a_i$$

(b) If the set S is uncountable, prove that

$$\sum_{s \in S} a_s = +\infty$$

unless $a_s = 0$ for all but countably many s.

2. Complex Analysis

(1) (a) Prove Liouville's theorem that a bounded entire function must be constant.
(b) Use this to prove the Fundamental Theorem of Algebra.

(2) (a) The open unit disk centered at zero in $\mathbb{R}$ is the interval $(-1,1)$. Construct a infinitely differentiable bijective function $f:(-1,1)\to\mathbb{R}$ such that the inverse $f^{-1}:\mathbb{R}\to(-1,1)$ is also infinitely differentiable.

(b) Show that the analogous statement fails for the complex numbers, i.e. there is no holomorphic bijection $f: D\to\mathbb{C}$ with holomorphic inverse. Here D is the open unit disk centered at zero in $\mathbb{C}$.

(3) Let $f:\mathbb{C}\to\mathbb{C}$ be an entire function that has a power series expansion

$$f(z)=\sum_{n\geqslant 0} a_n z^n$$

for all $z\in\mathbb{C}$.

(a) Compute the contour integral

$$\frac{1}{2\pi i}\oint_{\gamma}\frac{f(z)}{z}dz$$

where γ is the unit circle oriented counterclockwise. Do not appeal to Cauchy's theorem. You may use the fact that the power series expansion converges uniformly on open disks.

(b) For $k>1$, compute the contour integral

$$\frac{1}{2\pi i}\oint_{\gamma}\frac{f(z)}{z^k}dz$$

Again, do not appeal to Cauchy's theorem. You may use the fact that the power series expansion converges uniformly on open disks.

(c) Let $f:\mathbb{C}^*\to\mathbb{C}$ be a holomorphic function such that for all $\theta\in\mathbb{R}$,

$$f(e^{i\theta})=\sin(\theta)$$

What is the value $f(z)$ for arbitrary $z\in\mathbb{C}$? Prove your answer.

(4) (a) Let $U=\{z\in\mathbb{C}\mid |z-1|<1\}$ be the open unit disk centered at 1. Prove that there exists a function $f:U\to\mathbb{C}$ such that $e^{f(z)}=z$ for all $z\in U$.

(b) Can we enlarge the domain where f is defined? Prove that f cannot be extended to a holomorphic function $\mathbb{C}^*\to\mathbb{C}$.

(c) What does it mean that f is a "multivalued function" on $\mathbb{C}^*$?

(d) As above, let $U=\{z\in\mathbb{C}\mid |z-1|<1\}$ be the open unit disk centered at 1. Prove that there exists a function $g:U\to\mathbb{C}$ such that $g(z)^2=z$.

(e) Can we enlarge the domain where g is defined? Prove that g cannot be extended to a holomorphic function $\mathbb{C}^*\to\mathbb{C}$.

(f) What does it mean that g is a "multivalued function" on $\mathbb{C}^*$?

(5) (a) Use the argument principle to count (with multiplicity) the number of zeros of the polynomial z^4+3z^3+z+10 in the upper-right quadrant

$$Q=\{z\in\mathbb{C}\mid \mathrm{Re}(z)>0 \text{ and } \mathrm{Im}(z)>0\}$$

(b) Use the residue theorem to compute the integral

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx$$