Instructions. Answer question 1 (below) and at least six additional questions, with at least three coming from part **A** (combinatorics). Those being tested in two additional areas should answer at least three questions from parts **B** and **C**, with at least one question coming from each part. Those being examined in only one additional area should answer at least two questions from the other relevant part (**B** or **C**) of the exam, choosing the sixth question either from part **A** or from the other relevant part. Please do not choose a question if your answer to it would closely duplicate your answer to question 1.

You may answer more questions if you have time, but careful solutions to fewer problems will be preferable to careless solutions to more. Also: should you find yourself at an impasse, it is far better to admit this than it is to fudge. Doing the latter only leaves the impression that you don't realize that what you're writing down is wrong.

Question 1. Choose for discussion one important theorem you have learned. Give the background, explaining why the theorem is interesting and important. Then state and prove the theorem. If it makes sense to do so, you may introduce the topic with a problem that the theorem can solve and use the theorem to solve the problem.

Part A Combinatorics and Graph Theory

A1. A subset S of the vertices of a simple graph G is called *independent* if no two vertices of S are adjacent in G. The *independence number* of G, $\alpha(G)$, is then the maximum of the sizes of the independent sets. Show that if G has n vertices, then

$$\alpha(G) \cdot \chi(G) \geq n,$$

where $\chi(G)$ is the chromatic number of G.

A2. Let c_n be the number of ways to distribute n (indistinguishable) pieces of candy among three children in such a way that the oldest child receives an even number of pieces, the middle child receives an odd number of pieces, and the youngest child receives a number of pieces that is congruent to 2 (mod 3). Find a closed form for the generating function

$$F(x) = \sum_{n=0}^{\infty} c_n x^n,$$

and find the radius of convergence.

A3. Let $(k_1, \ldots, k_n)$ be a sequence of $n \geq 1$ nonnegative integers such that $n = k_1 + 2k_2 + 3k_3 + \cdots + nk_n$. Prove that the number of partitions of an n–element set into

$$\begin{cases} k_1 & \text{1–element subsets} \\ k_2 & \text{2–element subsets} \\ k_3 & \text{3–element subsets} \\ \cdots & \\ k_n & n\text{–element subsets} \end{cases}$$

is

$$\frac{n!}{\prod_{t=1}^{n} \left[(t!)^{k_t} \cdot k_t!\right]}.$$

A4. Show that for any positive n and r:

$$S(n+1, r) = \sum_{k=0}^{n} \binom{n}{k} S(n-k, r-1),$$

where $S(x, y)$ is the Stirling number of the second kind. ($S(x, y) = 0$ for $x < y$.)

A5. Prove that the number of partititions of an integer n with at most two repeats of each part equals the number of partitions of n into parts none of which is divisible by three. For example, if $n = 4$, there are four partitions of each kind.

At most two repeats:	**No multiples of three:**
$4 = 4$	$4 = 4$
$4 = 3 + 1$	$4 = 2 + 2$
$4 = 2 + 2$	$4 = 2 + 1 + 1$
$4 = 2 + 1 + 1$	$4 = 1 + 1 + 1 + 1$

(*Hint*: The Principle of Inclusion–Exclusion is relevant.)

A6. You are going to make a 12–bead bracelet. Each bead in your supply is either spherical or cubical, and each is one of k different colors. Your bracelet will have eight spherical and four cubical beads, with every third bead being cubical. How many different bracelets can you make? (Two bracelets are identified as the same if you can get the second one by either rotating the first one or flipping it over.)

A7. There is a finite projective plane of order 2 (three points on each line; three lines through every point; a total of 7 lines and 7 points). The plane generates a graph: the vertices are the points, with an edge between two vertices/points iff they are on a common line. Show that this graph is not planar.

Part B Combinatorial Matrix Theory

B1. For $n, m \geq 2$, let G be a bipartite graph on the $n+m$ vertices

$$X \cup Y : X = \{x_1, \ldots, x_n\}, Y = \{y_1, \ldots, y_m\}, X \cap Y = \emptyset$$

such that all edges connect vertices of X to vertices of Y. Show that a largest set S of edges such that no two edges in S are incident at a common vertex has the same size as a smallest set of vertices

$$A \cup B : A \subseteq X \text{ and } B \subseteq Y$$

such that every edge of G connects a vertex of A to a vertex of B.

B2. Let A and B be nonnegative $n \times n$ matrices such that all lines in A sum to k and all lines in B sum to ℓ. Show that all lines in AB must sum to $k\ell$.

B3. Let $q = p^\alpha$, where p is prime and $\alpha \geq 1$. This problem outlines an independent proof that in the finite projective plane of order q constructed from the vector space

$$V = \mathrm{GF}(q) \times \mathrm{GF}(q) \times \mathrm{GF}(q),$$

the number of "lines" is $q^2 + q + 1$. Prove each statement.

[a]: The number of ordered pairs of independent vectors in V is $(q^3 - 1)(q^3 - q)$.

[b]: For any two-dimensional subspace $W \subseteq V$, the number of ordered bases of W is $(q^2 - 1)(q^2 - q)$.

[c]: The number of two-dimensional subspaces of V is $q^2 + q + 1$.

B4. Let A and B be $n \times n$ and $m \times m$ matrices respectively, with respective characteristic polynomials

$$\phi_A(x) = (x - \lambda_1) \cdots (x - \lambda_n) \text{ and } \phi_B(x) = (x - \mu_1) \cdots (x - \mu_m).$$

Show that the charactistic polynomial of $A \otimes B$ is

$$\phi_{A \otimes B}(x) = \prod_{i=1}^{n} \prod_{j=1}^{m} (x - \lambda_i \mu_j).$$

B5. Let A be an $n \times n$ matrix. Find necessary and sufficient conditions on the Jordan form of A so that the sequence $(A, A^2, A^3, \ldots)$ remains bounded but does not converge.

B6. Let $(X_1, \ldots, X_m)$ be a sequence of sets, and let $x_1 \in X_1$. Show that if for every choice $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ we have

$$|X_{i_1} \cup X_{i_2} \cup \cdots \cup X_{i_k}| \geq k + 1,$$

then $(X_1, \ldots, X_m)$ has an SDR in which x_1 represents X_1.

B7. Show how a $(4t) \times (4t)$ Hadamard matrix may be used to generate a (nonsymmetric) 2–(v, k, λ) design with $v = 4t$, $k = 2t$, and $\lambda = 2t - 1$.

B8. Let A be a square, nonnegative, irreducible matrix. Show that the following are equivalent.

[a]. k is the index of imprimitivity of A.

[b]. k is the smallest positive integer for which $\big[\exists m\big]\big[A^m(I + A + \cdots + A^{k-1}) > 0\big]$.

[c]. k is the smallest positive integer for which $\big[\exists N\big]\big[\forall m \geq N\big]\big[A^m(I + A + \cdots + A^{k-1}) > 0\big]$.

Part C Theory of Computation

C1. Let

$$M_1 = (K_1, \{a, b\}, \delta_1, s_1, F_1)$$

and

$$M_2 = (K_2, \{a, b\}, \delta_2, s_2, F_2)$$

be two DFA's. Construct from them a DFA M_3 for which $L(M_3) = L(M_1) \cap L(M_2)$. Prove that your DFA works.

C2. Co–$\mathcal{NP}$ is the set of languages

$$\{L \subseteq \Sigma^* : \overline{L} \in \mathcal{NP}\}.$$

Suppose one had a language L_0 that was both $\mathcal{NP}$–complete and in co–$\mathcal{NP}$. Show that this would imply that $\mathcal{NP} =$ co–$\mathcal{NP}$.

C3. Show that the following context-free grammar generates the language

$$\big\{w \in \{a, b\}^* : w \text{ has the same number of } a\text{'s and } b\text{'s } \big\}.$$

$V = \{S, A, B\}$; $\Sigma = \{a, b\}$; and the rules are listed below.

$$S \longrightarrow aB \bigm| bA \bigm| \epsilon$$

$$A \longrightarrow aS \bigm| bAA$$

$$B \longrightarrow bS \bigm| aBB$$

C4. Show that the function

$$f(x) = x^{x^{\cdot^{\cdot^{\cdot^{x}}}}} \quad (x \text{ times})$$

is primitive recursive.

C5. Say you have a CFG G in Chomsky Normal Form (so that every rule has exactly two symbols to the right of the "$\longrightarrow$"). Describe an algorithm for deciding whether $L(G)$ is finite or not.

C6. Show that the language $\{a^n b^{n^2} : n = 1, 2, 3 \ldots\}$ is not context free.

C7. Let $L \subseteq \Sigma^*$ be a language such that both L and $\overline{L}$ are r.e. Show that L is recursive.

C8. Describe a Turing machine that computes the function $f(w) = ww^R$, for $w \in \{a, b\}^*$.