The four problems have 12 parts, two of which are quite challenging. Answer at most 10. Do not linger on parts you find easy. Try to answer harder parts with precision and insight.

1. Patrol officers are to be paired for the 3–11 pm shift. Each pair must have an experienced and an inexperienced officer who are compatible. Compatible pairs are indicated below.

Experienced \ Inexperienced	1	2	3	4	5	6	7	8	9	10
1		c	c							
2			c				c			c
3		c				c				
4		c								
5	c		c	c	c				c	
6						c		c	c	
7				c		c			c	
8								c		
9			c		c		c			c
10				c		c		c		

(a) What is the largest possible number of pairs with each officer in at most one pair?

(b) Prove that your answer to (a) is correct. Your proof should, of course, be concise.

(c) If the compatibility table were replaced by a ratings matrix C with nonnegative real entries and the goal were to maximize the sum of the ratings of the formed pairs, why is there a best pairing among the optimal solutions to the following problem?

$$\begin{aligned}
\text{Maximize } & \sum_{i,j} c_{ij}x_{ij} \\
\text{subject to } & \sum_{j} x_{ij} = 1, \text{ for } i = 1, \ldots, 10 \\
& \sum_{i} x_{ij} = 1, \text{ for } j = 1, \ldots, 10 \\
& x_{ij} \geq 0, \text{ for all } (i,j).
\end{aligned}$$

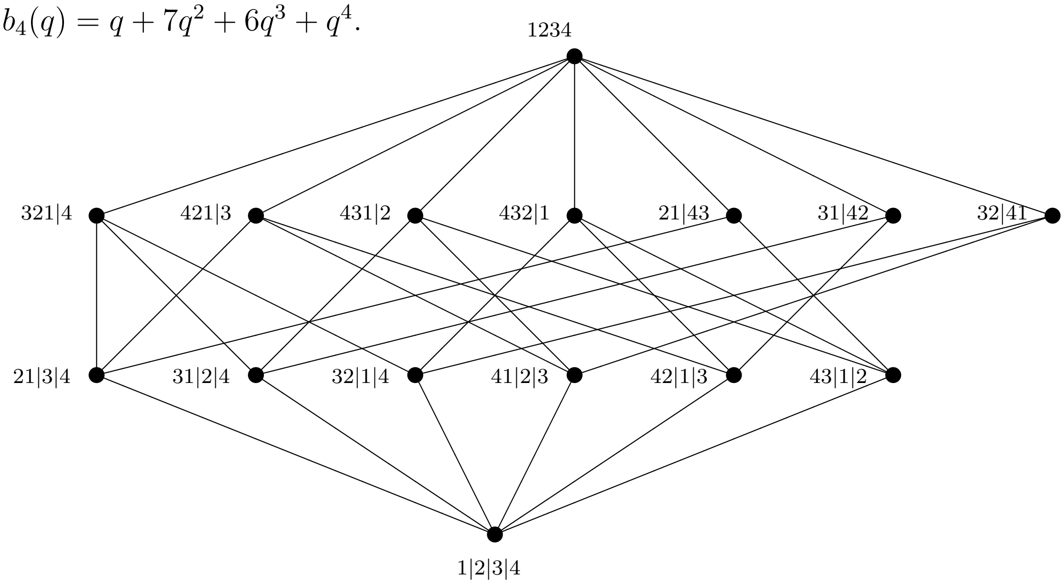
2. A nonincreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of nonnegative integers whose sum $|\lambda|$ is finite is called a *partition* of n if $|\lambda| = n$. We call λ_i a *part* of λ if $\lambda_i > 0$. Prove the following facts. At least one of your proofs should use a generating function and at least one should use a bijection.

(a) The number of partitions of n into distinct parts equals the number into odd parts.

(b) The number of partitions of n into odd parts of which exactly k are distinct equals the number into distinct parts in which exactly k sequences of consecutive positive integers occur. (For example, $(11, 11, 9, 5, 5, 5, 0, \ldots)$ has exactly 3 distinct odd parts and in $(11, 10, 9, 8, 6, 2, 0, \ldots)$ exactly 3 sequences of consecutive positive integers occur.)

(c) The number of partitions of n into distinct parts such that $\sum_{i\geq 1}(-1)^{i-1}\lambda_i = k$ equals the number into k odd parts.

3. Let $n \geq 0$ be an integer, and let $[n] = \{1, 2, \ldots, n\}$ (so that $[0] = \emptyset$). A *set partition* of $[n]$ is a collection of nonempty disjoint subsets of $[n]$, called *blocks*, whose union is $[n]$. The number of set partitions of $[n]$ is called the n^{th} *Bell number* b_n. The number of blocks in a set partition π is denoted $|\pi|$. The n^{th} *Bell polynomial* is the polynomial $b_n(q) = \sum_\pi q^{|\pi|}$, where the sum is over set partitions π of $[n]$ (so that $b_n(1) = b_n$). For example, $b_4 = 15$ and $b_4(q) = q + 7q^2 + 6q^3 + q^4$.



(a) Find $\sum_{n\geq 0} b_n \frac{x^n}{n!}$.

(b) Prove that the sequence of Bell numbers is *log-convex*: ${b_n}^2 \leq b_{n-1}b_{n+1}$ for $n \geq 1$. Does the fact that $b_k b_\ell \leq b_{k-1}b_{\ell+1}$ for $k \leq \ell$ follow?

(c) Prove that the sequence of Bell polynomials is *q-log-convex*: $b_{n-1}(q)b_{n+1}(q) - b_n(q)^2$ has nonnegative coefficients for $n \geq 1$. Does the fact that $b_{k-1}(q)b_{\ell+1}(q) - b_k(q)b_\ell(q)$ has nonnegative coefficients for $k \leq \ell$ follow?

4. Let τ, μ and ν be partitions such that $|\tau| = |\mu| + |\nu|$. Let $f(\tau, \mu, \nu)$ be the number of functions T with domain $\{(i,j) \,|\, \mu_i < j \leq \tau_i\}$ such that the number of pairs (i,j) that T maps to k is ν_k, for $k \geq 1$, and such that

- $T(i,j) \leq T(i,j')$ for $j < j'$;
- $T(i,j) < T(i',j)$ for $i < i'$;
- $n_k(I,J) \geq n_{k+1}(I,J)$ for $I \geq 1$, $\mu_I \leq J < \tau_I$ and $k \geq 1$,

  where $n_k(I,J) = \#\left\{ (i,j) \;\middle|\; T(i,j) = k \text{ and } \begin{array}{l}\text{either } i < I \text{ and } \mu_i < j \leq \tau_i \\ \text{or } i = I \text{ and } J < j \leq \tau_I\end{array} \right\}$.

(a) What is the algebraic significance of the numbers $f(\tau, \mu, \nu)$?

(b) Why does $f(\tau, \nu, \mu)$ equal $f(\tau, \mu, \nu)$?

(c) What is $f(\tau, \mu, \nu)$ if $\mu = (0, 0, \ldots)$ and $\nu = (1, \ldots, 1, 0, \ldots)$?