Swarthmore College Honors Exam - Spring 2009
Combinatorics and Combinatorial Representation Theory

Instructions:

- There is more here than you will have time to complete. First, look over the problems. Then, carefully select problems that you choose to answer. In addition to picking problems that you can answer well, please try to spread out your selections to cover different topics from both courses.
- There are three parts:
  (I) Exercises
  (II) Proofs
  (III) Overview of the subjects.

  This is a 3 hour exam. Please try to spend about 60 to 90 minutes on part (I), 60 to 90 minutes on part (II) and 30 minutes on part (III).

  After the appropriate time on a section, stop and move on even though you could continue.
- Write your answers clearly and concisely. Good style in writing mathematics is important. If there is part of an answer that you can't do well it is better admit the gap then to make a mistake or fudge your answer. Then move on to the parts you can do well.
- This is closed book, notes etc.
- The problems vary in length, so it is hard to predict the portion that you might answer. Do as much as you can, taking care not to ponder any one problem for too long.
- Try to pick problems that will allow you show off your mathematical skills.

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(I) Exercises

The questions in this section are designed to check if you know basic results and can do elementary computations. None of the answers should be very long. Some problems require proofs, outlines of proofs or explanations. These should be fairly short. You do not need to put in every detail. Instead do enough to show the main ideas. For more computational problems at least show some of your work and give short explanations of what you are doing when appropriate.

1: For each of the following give a short proof/explanation of your answer.
(a) How many k element multisets of $\{1, 2, \ldots, n\}$ are there? That is, how many non-negative integral solutions to $\sum_{i=1}^{n} x_i = k$ are there?
(b) How many non-negative integral solutions to $\sum_{i=1}^{n} x_i = k$ are there such that the x_i satisfy $x_i \geq r$ for a given r?
(c) Use generating functions or inclusion-exclusion to determine an expression for the number of non-negative integral solutions to $\sum_{i=1}^{n} x_i = k$ such that $x_i \leq s$ for a given s.

2: Give combinatorial arguments for the following recurrences (for $n \geq 1$).
Here $P(n, k)$ denotes the number of k permutations of $\{1, 2, \ldots, n\}$,
the (signless) Stirling numbers of the first kind $\left[\begin{array}{c} n \\ k \end{array} \right]$ count the number of permutations of $\{1, 2, \ldots, n\}$ with exactly k cycles and
the Stirling numbers of the second kind $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ count the number of partitions of $\{1, 2, \ldots, n\}$ into exactly k parts.

(a) $P(n, k) = P(n-1, k) + kP(n-1, k-1)$
(b) $\left[\begin{array}{c} n \\ k \end{array} \right] = (n-1) \left[\begin{array}{c} n-1 \\ k \end{array} \right] + \left[\begin{array}{c} n-1 \\ k-1 \end{array} \right]$
(c) $\left\{ \begin{array}{c} n \\ k \end{array} \right\} = k \left\{ \begin{array}{c} n-1 \\ k \end{array} \right\} + \left\{ \begin{array}{c} n-1 \\ k-1 \end{array} \right\}$

3: The Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ count many things. For example,
- The number of different ways to place $n-1$ pairs of parentheses in the product $x_0 x_1 \ldots x_n$
- The number of plane binary trees with $n+1$ leaves. These are rooted trees where where symmetry is not used (mirror images are considered distinct) and each non leaf vertex has exactly 2 children.
- The number modified ballot lists of length $2n$. These are sequences of $+1$'s and -1's (n of each type) such that every initial segment has at least as many ones as negative ones. That is, it is a list $a_1, a_2, \ldots, a_{2n}$ of $+1$'s and -1's such that $\sum_{i=1}^{k} a_i \geq 0$ with equality when $k = 2n$.
- Standard Young tableaux with two rows each of length n. These are $2 \times n$ arrays of the numbers $1, 2, \ldots, 2n$ arranged to be increasing along each row and column.

(a) Prove directly, using some interpretation of the Catalan numbers that there are indeed C_n such objects.
(b) Pick two interpretations of Catalan numbers (either those above or other that you know of) and describe a bijection between the two sets.

4: **(a)** Prove that every tree with maximum degree $\Delta > 1$ has at least Δ leaves (vertices with degree 1).
(b) Construct a 17 vertex tree with maximum degree 6 having exactly 6 leaves.
(c) Show that the result in part (a) is best possible. That is show that for each choice of n, Δ with $n > \Delta \geq 2$ there exists a tree with n vertices, maximum degree Δ and exactly Δ leaves.
(d) In part (c) why did we assume that $n > \Delta$? Could there be such a tree with $n \leq \Delta$? Why or why not?

5: **(a)** State Kuratowski's Theorem for planar graphs.
(b) Use Euler's formula and its consequences to prove that K_5 and $K_{3,3}$ are not planar.
(c) A graph is outerplanar if it has a planar embedding with all vertices on the same (outer) face. Prove that G is outerplanar if and only if it has no subgraph that is a subdivision of K_4 or $K_{2,3}$.
(d) Prove that an outerplanar graph is 2 face colorable. (Recall that an outerplanar graph has an embedding on the plane with every vertex on the outer face.)
(e) Without using the four color theorem prove that a Hamiltonian planar graph is four face colorable. (A graph is Hamiltonian if it has a cycle through all of the vertices.)

6: Determine a stable matching using the proposal algorithm with preferences as follows (explain the process of what you are doing, do not just give an answer):

Men $\{v, w, x, y, z\}$	Women $\{a, b, c, d, e\}$
$v : a > b > c > e > d$	$a : z > w > y > v > x$
$w : a > b > c > e > d$	$b : y > z > w > x > v$
$x : c > a > d > b > e$	$c : v > x > w > y > z$
$y : c > d > a > b > e$	$d : v > x > w > y > z$
$z : c > d > b > a > e$	$e : z > w > y > v > x$

7: **(a)** Prove that two permutations are in the same conjugacy class if and only if they have the same cycle type.
(b) What is the size of the conjugacy class of cycles of type $(1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$?

8: **(a)** Consider the defining representation of S_n, $X(\pi)$ with $x_{i,j} = 1$ if $\pi(j) = i$ and $x_{i,j} = 0$ otherwise. Show that this is a representation.
(b) Show that the character $\chi(\pi)$ is the number of fixed points of π.

9: Let $sl_3(\mathbb{C}) = \{A \in Mat_3 | Tr(A) = 0\}$ (i.e., the space of all complex 3×3 with trace, the sum of diagonal entries, equal to 0). One basis is $\{E_{11} - E_{22}, E_{22} - E_{33}\} \cup \{E_{ij} | i \neq j\}$ where E_{ij} has all entries 0 except a 1 in row i, column j. Define an action of S_3 on sl_3 by $\omega A \omega^{-1}$, where ω is a permutation matrix.
(a) Show that this defines an S_3 module.
(b) Find the 8×8 representation matrices of $\tau_1 = (1, 2)$ and $\tau_2 = (2, 3)$ (the permutations that transpose the first and second elements and second and third elements respectively).

10: **(a)** Consider the permutation (in two line notation) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 4 & 9 & 5 & 7 & 1 & 2 & 8 & 3 \end{pmatrix}$. Construct the sequence of tableaux generated by the Robinson-Schensted algorithm.
(b) The Robinson-Schensted algorithm establishes a bijection between permutations and certain pairs of tableaux. This then can be used to prove an identity involving the f^λ, the number of standard tableaux of shape λ and $n!$, the number of permutations. State this identity and prove it.
(c) In terms of representations of the symmetric group why do we care about the numbers f^λ?

11: **(a)** Use the recursions of problem 2 to generate tables for Stirling numbers of the first and second kinds for values of n and k in the range $0, 1, 2, \ldots, 6$.
(b) Use the values from the tables in part (a) to fill in the blanks in
$\binom{x}{5} = \underline{\quad}\ x^5 + \underline{\quad}\ x^4 + \underline{\quad}\ x^3 + \underline{\quad}\ x^2 + \underline{\quad}\ x$
and
$x^5 = \underline{\quad}\ \binom{x}{5} + \underline{\quad}\ \binom{x}{4} + \underline{\quad}\ \binom{x}{3} + \underline{\quad}\ \binom{x}{2} + \underline{\quad}\ \binom{x}{1}$.

12: **(a)** Use the hook length formula to determine $f^{(4,3,2)}$ and $f^{(6^2)}$. That is, the number of standard Young tableaux with shapes $(4, 3, 2)$ and $(6^2) = (6, 6)$.
(b) Determine $f^{(n^2)}$. That is, determine the number of standard Young tableaux with two rows, each of length n.

13: Describe the group algebra of a group G and explain why it is a G-module.

(II) Proofs

For questions in this section show that you can write clear and concise proofs. A few of the questions are more conceptual about the ideas related to proofs. If you choose to omit some details state that you are doing this. For problems with multiple parts you may use results of previous parts even if you did not answer them.

14: Recall that the Stirling numbers of the first kind $\left[\begin{matrix} n \\ k \end{matrix}\right]$ count the number of permutations of $\{1, 2, \ldots, n\}$ with exactly k cycles and the Stirling numbers of the second kind $\left\{\begin{matrix} n \\ k \end{matrix}\right\}$ count the number of partitions of $\{1, 2, \ldots, n\}$ into exactly k parts. Also, $x^{\overline{n}} = x(x+1)\cdots(x+n-1)$ and $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$.

(a) Show *one* of the following:

$x^{\overline{n}} = \sum_k \left[\begin{matrix} n \\ k \end{matrix}\right] x^k \quad \text{or} \quad x^{\underline{n}} = \sum_k \left[\begin{matrix} n \\ k \end{matrix}\right] (-1)^{n-k} x^k$

(b) Show *one* of the following:

$x^n = \sum_k \left\{\begin{matrix} n \\ k \end{matrix}\right\} x^{\underline{k}} \quad \text{or} \quad x^n = \sum_k \left\{\begin{matrix} n \\ k \end{matrix}\right\} (-1)^{n-k} x^{\overline{k}}$

(c) Show that Stirling numbers of the first kind and second kind are 'inverses' in *one* of the following ways:

$$\sum_k \left[\begin{matrix} n \\ k \end{matrix}\right] \left\{\begin{matrix} k \\ m \end{matrix}\right\} (-1)^{n-k} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}$$

or

$$\sum_k \left\{\begin{matrix} n \\ k \end{matrix}\right\} \left[\begin{matrix} k \\ m \end{matrix}\right] (-1)^{n-k} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}$$

15: Let G be a connected plane graph with n vertices, e edges and f faces.

(a) Prove that $n - e + f = 2$ (Euler's Theorem).

(b) Use part (a) to prove that a simple planar graph with $n \geq 3$ satisfies $e \leq 3n - 6$.

(c) Use part (b) to prove that a planar graph has a vertex with degree at most 5.

(d) Outline (you do not need to provide all of the details) a proof that every planar graph has chromatic number at most 5. You may use part (c).

(e) Explain what goes wrong in the proof of part (d) when you attempt to prove the 4 color theorem.

16: Prove *one* of the following:

(a) The Gale-Shapley proposal algorithm produces a stable matching.

(b) If man x is paired with women a in some stable matching then a does not reject x in the Gale-Shapely proposal algorithm (with men proposing to women). This shows that among all stable matchings every man is happiest with the matching produced by the algorithm.

17: **(a)** Prove the infinite version of Ramsey's Theorem: Let G be the complete infinite graph with vertices $V = \{v_i | i \in \mathbb{N}\}$. Given any 2-coloring of the edges, G will contain an infinite complete monochromatic subgraph.
(b) Prove the finite version of Ramsey's Theorem. For each $n \in \mathbb{N}$ then is an $m \in \mathbb{N}$ such that $R(n, n) = m$ (every two coloring of the complete graph on m vertices contains a complete monochromatic subgraph on n vertices). You may use part (a) if you want but can give a direct proof if you prefer.

18: Write $e_k(n) = e_k(x_1, x_2, \ldots, x_n)$ for elementary symmetric functions and $h_k(n) = h_k(x_1, x_2, \ldots, x_n)$ for homogeneous symmetric functions. Replacing the variables with numbers indicates an evaluation of the function at these values. For example $e_{n-k}(1, 2, \ldots, n-1)$ is $e_{n-k}(x_1, x_2, \ldots, x_{n-1})$ with $x_i = i$ for $i = 1, 2, \ldots, n-1$.
(a) Show that $e_k(n) = e_k(n-1) + x_n e_{k-1}(n-1)$ and $h_k(n) = h_k(n-1) + x_n h_{k-1}(n)$ for $n \geq 1$.
(b) Show $\binom{n}{k} = e_k(1, 1, \ldots, 1) = h_k(1, 1, \ldots, 1)$ where there are n 1's for e_k and $(n-k+1)$ 1's for h_k.
(c) For the (signless) Stirling numbers of the first kind show $\left[\begin{array}{c} n \\ k \end{array} \right] = e_{n-k}(1, 2, \ldots, n-1)$
(d) For the Stirling numbers of the second kind show $\left\{ \begin{array}{c} n \\ k \end{array} \right\} = h_{n-k}(1, 2, \ldots, k)$

19: **(a)** Consider the two tableaux $P(\pi) = \begin{array}{ccccc} 1 & 2 & 3 & 4 & 9 \\ 5 & 7 & 8 & & \\ 6 & & & & \end{array}$ and $Q(\pi) = \begin{array}{ccccc} 1 & 3 & 5 & 6 & 7 \\ 2 & 4 & 9 & & \\ 8 & & & & \end{array}$.
Construct Viennot's shadow diagram and use this to recover the permutation π. Explain what you are doing.
(b) Use Viennot's construction to prove that taking inverses interchanges the corresponding Robinson-Schensted tableaux.
(c) Viennot's construction establishes a bijection between involutions (permutations π with $\pi^{-1} = \pi$) and certain pairs of tableaux. Give this bijection and outline a proof. The bijection can be used to prove an identity involving the f^λ, the number of standard tableaux of shape λ and the number of involutions in S_n. State and prove this identity.

20: Prove that the following are bases for Λ^n, the space spanned by monomial symmetric functions of degree n:
(a) $\{p_\lambda : \lambda \vdash n\}$ (power sum symmetric functions).
(b) $\{e_\lambda : \lambda \vdash n\}$ (elementary symmetric functions).
(c) $\{h_\lambda : \lambda \vdash n\}$ (complete homogeneous symmetric functions).

(III) Overview of the subjects.
Answer all three parts here.

21: You find yourself at a graduation party sitting with your father, your grandmother and one of your professors. They ask the following questions. What are your answers?
(a) From your father: We just spent all of this money for you to study combinatorics and combinatorial representation theory. What is it good for?
(b) From your grandmother: You just spent a lot of time studying combinatorics and combinatorial representation theory. What are they?
(c) From your professor: I just spent a lot of time trying to help you understand combinatorics and combinatorial representation theory. What results did you find really interesting and what did you think was a waste of time?

Clarifications:
(a) What do you say to someone who asks what the 'purpose' of (this particular area of) mathematics is. This might include potential applications that you know of and/or why its worthwhile even without applications.
(b) Try to think of one or two ideas or examples that you can describe in a few minutes without using too much technical language. These should illustrate or give some hint of what is being studied in these topics.
(c) State and give a sketch of a proof of some result that you found particularly interesting. Say something about why you liked it. Also, point out something that you did not like and explain why. This might be because it was too complicated or because for some reason you thought the problem was not interesting or perhaps something else. (Saying that you hated topic X because the week that it was covered you also had a huge history paper is probably not a good answer.)