Swarthmore College
Combinatorics and Combinatorial Optimization Honors Exam
Thursday May 16, 1996

Instructions: This exam has three parts. Part I consists of elementary problems and part II includes problems that are more theoretical in nature. In part III the problems are more open ended, try to give a concise and carefully thought out discussion of the issues.

Answer the questions carefully, showing necessary work. Careful answers to most of the questions are more important than careless answers to all of them. However, be sure to answer *some questions from each part,* so do not spend to much time working on any one part. Make sure that you leave plenty of time for the discussions in part III.

Note - we will use the notation $\binom{n}{k}$ for binomial coefficients. Another notation for the same coefficient is $C(n,k)$.

Part I:

1. Suppose that we have k distinct objects. If we allow repetition (selecting multiple copies of an object) prove that the number of ways of selecting n objects (again, repeats are allowed) is $\binom{n+k-1}{k-1}$. How does this relate to the number of solutions using non-negative integers to $x_1 + x_2 + \cdots + x_k = n$ and why?

2. If a professor returns exams to a class of 5 students without regard to the names on the papers (i.e., randomly), how many different ways can the papers be handed out so that no student gets their own exam? How many ways if there are n students?

3. Give an example of a graph which has no cliques of size 3 and which is not 2-chromatic. Give an example of a graph which has no cliques of size 4 and which is not 3-chromatic.

4. Show that

$$\sum_{k=1}^{n} \frac{1}{k} \sim \log(n).$$

5. Prove that the following decision problem is in the class NP: Given a graph G, does it contain a Hamiltonian circuit?

6. Let $\alpha(G)$ denote the size of a largest independent set of vertices in the graph G and let $\chi(G)$ denote the chromatic number of G. Let n be the number of vertices in G. Show that $\alpha(G) \geq n/\chi(G)$.

7. If n is a power of 3 show that if

$$T(n) = 3T\left(\frac{n}{3}\right) + 5n$$

then $T(n) = \theta(n \log n)$.

8. Which of the following statements are correct? In each case explain your answer.

(a) If we say that 3-satisfiability is polynomially reducible to vertex cover then, since 3-satisfiability is NP-complete this will show that vertex cover is NP-complete.

(b) If we say that 3-satisfiability is polynomially reducible to vertex cover then, since vertex cover is NP-complete this will show that 3-satisfiability is NP-complete.

Part II:

1. The Fibonacci numbers are given by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$.

(a) Find the generating function for the sequence F_n of Fibonacci numbers.

(b) Give an explicit formula for F_n.

2. What is the generating function for the sequence $\{q^{\neq}(n)\}$ where $q^{\neq}(n)$ is the number of partitions of n into unequal parts (i.e., parts whose sizes are all different)? What is the generating function for the sequence $\{odd(n)\}$ where $odd(n)$ is the number of partitions into parts whose sizes are odd numbers? Use this to prove that $odd(n) = q^{\neq}(n)$.

3. Find the pattern inventory (in Polya's Theorem) for coloring a 2×2 chessboard with two colors.

4. (a) For the Ramsey numbers show that $R(k,l) \leq R(k-1,l) + R(k,l-1)$ and use this inequality to show that

$$R(k,l) \leq \binom{k+l-2}{k-1}$$

for $k, l \geq 2$.

(b) Use Stirling's approximation and part (a) to get an asymptotic estimate on an upper bound for $R(n+1, n+1)$.

5. Sketch a proof of Euler's Theorem: If $(x,m) = 1$ (i.e., $gcd(x,m) = 1$) then $x^{\phi(m)} \equiv 1 \pmod{m}$ where $\phi(m)$ is Euler's phi function (totient).

6. Show that for any real constants a and b where $b > 0$

$$(n+a)^b = \Theta(n^b).$$

Part III:

1. What is mathematical induction? Give an informal description, one that you might give to a friend with little or no mathematical background in mathematics as well as a more formal description. Illustrate by giving an example of a simple mathematical proof using induction (other than one in parts I or II). This can be any example, pick your 'favorite.'

2. Describe the relationship between the binomial coefficients, the binomial theorem and Pascal's triangle. How is the interpretation of $\binom{n}{k}$ (n choose k) as the number of ways to pick k objects from a set of size n involved? How is this all related to the number of subsets of a set?

3. Give a general description Burnside's Theorem and Polya's Theorem. How are the two theorems related? What sort of problems are they meant to solve? What methods are employed? This does not need to be a detailed description, rather an explanation of the ideas involved. You may wish to illustrate with a small example.

4. What is an efficient algorithm in the formal mathematical sense? In practice will an efficient algorithm always be 'better' than an non-efficient algorithm for the same problem? Why or why not?

5. Give an informal description of the theory of NP-completeness, the sort you might give to a friend with little or no mathematical background. Include a description of what an NP problem is and what an NP-complete problem is. How does this relate to decision problems and to optimization problems?