HONORS EXAM IN ANALYSIS

(1) Let $\{x_n\}$ denote a sequence of real numbers.
   (a) State what it means for the sequence $\{x_n\}$ to converge to a real number x.
   (b) Suppose that a sequence $\{x_n\}$ is bounded and for all $n \in \mathbb{N}$ $x_n \leq x_{n+1}$. Show that the sequence must converge.

(2) Let X and Y denote subsets of $\mathbb{R}$. Let $f : X \to Y$.
   (a) State what it means for f to be continuous at a point $x \in X$.
   (b) State what it means for f to be a uniformly continuous function on X.

(3) Determine which of the following statements are true and which are false. Provide counter-examples for the false statements and prove the true ones. In the case of a false statement make sure you thoroughly explain why the example you give is a counter-example to the statement. In this problem X and Y will denote subsets of $\mathbb{R}$.
   (a) If a function $f : \mathbb{R} \to \mathbb{R}$ is continuous then for all open subsets O of $\mathbb{R}$, $f^{-1}(O)$ is also open.
   (b) If a function $f : \mathbb{R} \to \mathbb{R}$ is continuous and onto then for all open subsets O of $\mathbb{R}$, $f(O)$ is also open.
   (c) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and C is a compact subset of $\mathbb{R}$, then $f(C)$ is also compact.
   (d) If $f : X \to Y$ is continuous, one-to-one and onto, and if X is a compact subset of $\mathbb{R}$, then for all closed subsets A of X, $f(A)$ is a closed subset of Y.

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(4) Let $\{f_n\}$ be a sequence of functions $f_n : X \to Y$, where X and Y are metric spaces.

(a) State what it means for the sequence $\{f_n\}$ to converge (pointwise) to a function f.

(b) State what it means for the sequence $\{f_n\}$ to converge uniformly to a function f.

(5) Let $C[a, b]$ denote the set of continuous functions $f : [a, b] \to \mathbb{R}$ and for $f \in C[a, b]$ set

$$\|f\| = \sup_{x \in [a,b]} |f(x)|.$$

Here the notation sup, short for supremum, means the same thing as least upper bound.

(a) Show that the function $\rho : C[a, b] \times C[a, b] \to \mathbb{R}$ given by

$$\rho(f, g) = \|f - g\|$$

is a metric.

(b) Show that if a sequence of functions $\{f_n\}$ in $C[a, b]$ converges to $f \in C[a, b]$ in this metric then it converges uniformly.

(c) Is it possible for a sequence of functions $\{f_n\}$ in $C[a, b]$ to converge to a function f in this metric and $f \notin C[a, b]$?

(6) Let $S \subset \mathbb{R}^n$.

(a) State the definition of a rectifiable set S.

(b) State the definition of a set S of measure zero.

(c) Prove that S is rectifiable if and only if it is bounded and the boundary of S has measure zero.

(d) Show by example that not all bounded open subsets of $\mathbb{R}^n$ need be rectifiable.

(7) Let A be an $n \times n$ matrix. Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation $h(x) = A \cdot x$. Let S be a rectifiable set in $\mathbb{R}^n$ and denote its volume by $v(S)$. Let $T = h(S)$.

Show that

$$v(T) = |detA| \cdot v(S).$$

(8) Let $f : [a, b] \to \mathbb{R}$, with $a < b$.

(a) Prove Rolle's Theorem:
Suppose f is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then $f'(x_0) = 0$ for some $x_0 \in (a, b)$.

(b) Prove by example that the condition of *differentiable on* (a, b) cannot be relaxed even at one point for Rolle's Theorem to hold.