HONORS EXAM IN ANALYSIS

(1) Let $\{x_n\}$ denote a sequence of real numbers.
   (a) State what it means for the sequence $\{x_n\}$ to converge to a real number x.
   (b) Suppose that a sequence $\{x_n\}$ is bounded and for all $n \in \mathbb{N}$ $x_n \leq x_{n+1}$. Show that the sequence must converge.

(2) Let X and Y denote subsets of $\mathbb{R}$. Let $f : X \to Y$.
   (a) State what it means for f to be continuous at a point $x \in X$.
   (b) State what it means for f to be a uniformly continuous function on X.

(3) Determine which of the following statements are true and which are false. Provide counter-examples for the false statements and prove the true ones. In the case of a false statement make sure you thoroughly explain why the example you give is a counter-example to the statement. In this problem X and Y will denote subsets of $\mathbb{R}$.
   (a) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $\{x_n\}$ is a sequence of real numbers which converges to $x \in \mathbb{R}$ then $\{f(x_n)\}$ converges to $f(x)$.
   (b) If $f : X \to Y$ is continuous and X is a closed subset of $\mathbb{R}$ then f is uniformly continuous.
   (c) If $f : X \to Y$ is continuous and X is a bounded subset of $\mathbb{R}$ then f is uniformly continuous.
   (d) If $f : X \to Y$ is continuous then
$$\{x : f(x) = 0\}$$
is a closed subset of X.
   (e) If $f : X \to Y$ is continuous and X is a closed subset of $\mathbb{R}$ then
$$\{x : f(x) = 0\}$$
is a closed subset of X.

Date: Summer, 2005.

(4) Let $C[a, b]$ denote the set of continuous functions $f : [a, b] \to \mathbb{R}$ and for $f \in C[a, b]$ set

$$\|f\| = \sup_{x \in [a,b]} |f(x)|.$$

Here the notation sup, short for supremum, means the same thing as least upper bound.

(a) Show that the function $\rho : C[a, b] \times C[a, b] \to \mathbb{R}$ given by

$$\rho(f, g) = \|f - g\|$$

is a metric.

(b) Is it possible for a sequence of functions $\{f_n\}$ in $C[a, b]$ to converge to a function f in this metric with $f \notin C[a, b]$?

(5) Let $\Sigma_{n=1}^{\infty} a_n$ denote an infinite series.

(a) State what it means for the series to be convergent.

(b) Prove that the series converges if and only if for every $\epsilon > 0$ there is an integer N such that

$$|\Sigma_{k=n}^{m} a_k| < \epsilon$$

if $m \geq n \geq N$.

(6) Prove that $exp(x) \geq 1 + x$ for all $x \in \mathbb{R}$.

(7) Let (X, f) be a dynamical system.

(a) State the definitions of attracting and repelling fixed points of (X, f).

(b) Identify the fixed points of (X, f) in the case that $X = \{0, 1\}^{\mathbb{Z}}$ and $f = \sigma$ and classify them as attracting, repelling, or neither.

(c) Give an example of a dynamical system with an attracting fixed point. Prove your claim.

(8) Let X be a metric space with metric ρ.

(a) State what it means for a subset of X to be dense.

(b) Give an example of a dynamical system (X, f) whose periodic points are dense. Prove your claim.

(c) Give an example of a dynamical system (X, g) whose periodic points are not dense. Prove your claim.