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HONORS EXAM IN REAL AND COMPLEX ANALYSIS

Answer as many of the following questions or parts of questions as you can. You may quote and use standard results but you need to fully explain your reasons. You may also use the statement of any part of a question in any subsequent part. Show all work and fully support all answers. Good luck.

1. Let f be uniformly continuous and nonnegative on $[0,\infty)$ and suppose that the improper Riemann integral of f on $[0,\infty)$ exists and is finite, that is, $\int_0^\infty f(x)\,dx < \infty$.

   (a) Show that $\lim_{x\to\infty} f(x) = 0$.

   (b) Does part (a) hold if the word "uniformly" is removed from the hypotheses on f? Prove or provide a counterexample.

2. Let f be a continuous real–valued function on $[0,1]$.

   (a) Show that $\sup_{x\in[0,1]} |f(x)| = M < \infty$ and that there is an $x_0 \in [0,1]$ such that $f(x_0) = M$.

   (b) Show that for every $\epsilon > 0$ there is a nonempty open interval $(a,b) \subseteq [0,1]$ such that $|f(x)| \geq M - \epsilon$ for all $x \in (a,b)$.

   (c) Show that for each $p > 1$, $\left(\int_0^1 |f(x)|^p\,dx\right)^{1/p} \leq M$ and that
   $$\lim_{p\to\infty}\left(\int_0^1 |f(x)|^p\,dx\right)^{1/p} = M.$$

3. A function f on $[0,1]$ is said to be of *bounded variationon* $[0,1]$ provided that there is a number M such that for every partition of $[0,1]$, $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$, $\sum_{i=1}^n |f(x_{i-1}) - f(x_i)| \leq M$. The function f is said to be *absolutely continuous* on $[0,1]$ provided that for every $\epsilon > 0$ there is a $\delta > 0$ such that for any collection of nonoverlapping subintervals of $[0,1]$, $\{[a_i,b_i] \subseteq [0,1]\}$, such that $\sum_i (b_i - a_i) < \delta$, $\sum_i |f(b_i) - f(a_i)| < \epsilon$.

   Let C denote the class of functions continuous at each point of $[0,1]$, UC the class of functions uniformly continuous on $[0,1]$, BV the class of functions of bounded variation on $[0,1]$ and AC the class of functions absolutely continuous on $[0,1]$.

   (a) Show that $C = UC$.

   (b) Show that $AC \subseteq BV$.

   (c) Show that $UC\backslash BV$ is nonempty, that is, find a function which is uniformly continuous on $[0,1]$ but not of bounded variation on $[0,1]$. (Hint: Consider a function of the form $x^\alpha \sin\left(\frac{1}{x^\beta}\right)$ for appropriately chosen α, $\beta > 0$.)

4. Let $\{x_n\}_{n=1}^{\infty}$ be a real–valued, bounded sequence.

(a) State the definition of $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$, and show that each exists as a finite number.

(b) Show that if x is an accumulation point of the sequence $\{x_n\}$ then

$$\liminf_{n\to\infty} x_n \le x \le \limsup_{n\to\infty} x_n.$$

5. Let $\{a_n\}_{n=1}^{\infty}$, $a_n \ge 0$ for all n, be increasing. Define the sequence $\{\sigma_n\}_{n=1}^{\infty}$, called the *sequence of arithmetic means*, by $\sigma_n = \dfrac{a_1 + a_2 + \cdots + a_n}{n}$.

(a) Show that $\{\sigma_n\}$ is a bounded sequence if and only if $\{a_n\}$ is bounded.

(b) Show that if $\lim_{n\to\infty} \sigma_n = L$, then $\lim_{n\to\infty} a_n = L$.

6. Show that $\displaystyle\int_0^{\pi/2} \frac{d\theta}{1+\sin^2(\theta)} = \frac{\pi}{2\sqrt{2}}$.

7. Suppose that the function $f(z)$ is analytic in a region R of the complex plane and let $f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$.

(a) Show that u and v are harmonic on R (now considered as a subset of the plane $\mathbf{R}^2$), that is, that $\Delta u = \Delta v = 0$ on R, where $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$.

(b) State what it means for a function on the complex plane to be a *conformal mapping.*

(c) Show that if $f(z)$ is analytic on R , and if $f'(z)$ does not vanish on R, then $f(z)$ is a conformal mapping.