ALGEBRAIC GEOMETRY EXAM

SPRING 2021

This exam consists of 9 problems. You are not required to solve them all, but rather should work on the ones you find interesting and approachable.

I am interested in seeing how you approach the various problems, so please turn in your solutions to a problem even if you can only make progress on some of the individual parts. If you find it useful, you may assume an earlier part of a problem when working on the later parts even if you haven't been able to solve it. However, please bear in mind that I would rather see substantial progress on a few problems than a handful of computations for every problem.

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(1) (a) Let k be $\mathbb{R}$ or $\mathbb{C}$. Show that $\mathbb{A}^2_k \setminus \{(0,0)\}$ is not an affine variety.
(b) Is $\mathbb{A}^2_k \setminus \{(0,0)\}$ an affine variety when k is a finite field? Why or why not?

(2) Let k be an algebraically closed field, and let $\mathbb{A}^n = \mathbb{A}^n_k$.
(a) Let J be the ideal $(xy, xz, yz) \subset k[x,y,z]$. Find $\mathrm{V}(J) \subset \mathbb{A}^3$. Is it irreducible? Is it true that $J = I(V(J))$?
(b) Let K be the ideal $(xy, (x-y)z)$. Find $\mathrm{V}(K)$, and calculate $\sqrt{K}$.

(3) Let k be an algebraically closed field. Define $\phi : \mathbb{A}^1_k \to \mathbb{A}^3_k$ by

$$\phi(t) = (t, t^2, t^3).$$

Let $W \subset \mathbb{A}^3_k$ be the image of ϕ. Prove that W is an irreducible algebraic variety.

(4) Let $\mathcal{C} = \mathrm{V}((x+y)(x-y)(x-2y))$ be a curve in $\mathbb{C}^2$.
(a) Show that $\mathcal{C}$ has a singular point at the origin.
(b) Describe the blow up of $\mathcal{C}$ at the origin (restricting your attention to the strict transform).
(c) How many points are there over the origin in the blow up you have described?

(5) Fix points $\mathbf{p} = [p_0, \ldots, p_n]$ and $\mathbf{q} = [q_0, \ldots, q_n]$ in $\mathbb{P}^n(\mathbb{C})$.
(a) Define a map $F : \mathbb{P}^1(\mathbb{C}) \dashrightarrow \mathbb{P}^n(\mathbb{C})$ by

$$F(u,v) = u\mathbf{p} - v\mathbf{q} = [up_0 - vq_0, \ldots, up_n - vq_n].$$

Describe the domain of F and show that F is one-to-one on its domain.
(b) Let $\ell = a_0x_0 + \cdots + a_nx_n$ be a linear homogeneous polynomial. Show that ℓ vanishes on the image of F if and only if $\mathbf{p}, \mathbf{q} \in \mathrm{V}(\ell)$.
(c) Fix $n = 3$ and let $\mathbf{p} = [1, -2, 0, 0]$ and $\mathbf{q} = [0, 1, 0, 0]$. Write the image of F as $\mathrm{V}(\ell_1, \ldots, \ell_k)$, where the ℓ_i are linear polynomials. How many do you need?

(6) Consider the *Cremona map* on $\mathbb{P}^2(\mathbb{C})$ given in projective coordinates by

$$[x, y, z] \mapsto [yz, xz, xy].$$

(a) Show that the Cremona map is a rational map. What is its domain?
(b) Show that the Cremona map is a birational map. (Hint: what happens if you compose the Cremona map with itself?)
(c) Sometimes the map on $\mathbb{P}^2(\mathbb{C})$ given in projective coordinates by

$$[x, y, z] \mapsto [1/x, 1/y, 1/z]$$

is also referred to as the Cremona map. Is this consistent terminology? Why or why not?

(7) Consider the *Segre map* $\Sigma_{m,n} : \mathbb{P}^m(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^{(m+1)(n+1)-1}$. One way to define the Segre map is to write coordinates of points in $\mathbb{P}^m(\mathbb{C})$ and $\mathbb{P}^n(\mathbb{C})$ as $(m+1)$- and $(n+1)$-dimensional column vectors, respectively, and to write coordinates of points in $\mathbb{P}^{(m+1)(n+1)-1}$ as $(m+1) \times (n+1)$ matrices. Then the Segre map is given by

$$\Sigma_{m,n}(\mathbf{p}, \mathbf{q}) = \mathbf{p}\mathbf{q}^T.$$

(a) Let $\mathcal{S} \subset \mathbb{P}^{(m+1)(n+1)-1}(\mathbb{C})$ be the image of the Segre map. One may define a projection map $\pi_1 : \mathcal{S} \to \mathbb{P}^m(\mathbb{C})$ by sending an $(m+1) \times (n+1)$ matrix to any of its nonzero columns. Prove that π_1 is well-defined (that is, its image does indeed lie in $\mathbb{P}^m(\mathbb{C})$ and the choice of a nonzero column doesn't matter).
(b) Fix any point $\hat{\mathbf{q}} \in \mathbb{P}^n$. Define a map $\phi : \mathbb{P}^m(\mathbb{C}) \to \mathbb{P}^m(\mathbb{C})$ by

$$\phi(\mathbf{p}) = \pi_1 \circ \Sigma_{m,n}(\mathbf{p}, \hat{\mathbf{q}}).$$

Show that ϕ is the identity map.

(8) Find all of the conics in $\mathbb{P}^2(\mathbb{C})$ which contain the points $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1]$, $[1, 1, 0]$, and $[1, -1, 0]$.

(9) Consider the elliptic curve E over $\mathbb{C}$ described by the affine equation $y^2 = x^3 - 5x$. Write $\oplus$ for the group law on this elliptic curve.
(a) Verify that $P = (-1, -2)$ and $Q = (0, 0)$ are points on E, and compute $P \oplus Q$.
(b) Let $\mathcal{O}$ be the identity element of the group law. Give an explicit description of $\mathcal{O}$.
(c) Find a point R such that $P \oplus R = \mathcal{O}$.