Swarthmore College
Department of Mathematics and Statistics
Honours Examination: Algebra
Spring 2026

On this exam, you should aim to solve **at least six problems, including at least two from each section**, as completely as possible. **Answers without clear justification will not receive full credit.** Once you are satisfied with your solutions to six problems, you are encouraged to make a second pass through the exam and attempt one additional problem.

When working on a problem, you may find it helpful to read through all of its parts before beginning. I am interested in your reasoning and your approach, so please submit your work even if it is incomplete. For problems with multiple parts, you are welcome to attempt later parts even if you have not completed earlier ones, provided that you clearly state any assumptions you make.

Section 1: Groups

1. Let G be a group of order 245.
   (a) Show that G has at most one subgroup of order 49.
   (b) Suppose that $H \leq G$. Show that for every $g \in G$, the set gHg^{-1} is a subgroup of G.
   (c) Let H be a unique subgroup of G of order k. Prove that $H \trianglelefteq G$.
   (d) Let H be a subgroup of order 49 of G. Using the previous parts, show that $H \trianglelefteq G$.

2. This problem concerns groups of order 8. In each part, justify your answer.
   (a) Are the groups $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/8\mathbb{Z}$ isomorphic? Explain your reasoning. Hence, determine how many abelian groups of order 8 there are up to isomorphism.
   (b) How many group homomorphisms are there from $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}/8\mathbb{Z}$?
   (c) How many group homomorphisms are there from $\mathbb{Z}/8\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$?

3. Let G be a group of order $45 = 3^2 \cdot 5$.

   (a) Determine all possible numbers of Sylow 5-subgroups and prove your answer.

   (b) Show that G has a normal Sylow subgroup.

   (c) Classify all groups of order 45 up to isomorphism.

4. (a) Let a finite group G act on a finite set X. For each $g \in G$, let X^g denote the set of elements of X that are fixed by g. Prove the following formula, known as Burnside's Lemma, for the number of orbits $|X/G|$ of this action:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

   (b) Use part (a) to compute the number of distinct colorings of the vertices of a square using 3 colors, up to rotation.

   (c) Repeat part (b), now including reflections.

Section 2: Rings and Fields

Note: Recall that, in general, a ring does not necessarily have a multiplicative identity.

5. Let $\varphi : \mathbb{Z}[x] \to \mathbb{Z}$ be defined by $p(x) \mapsto p(1)$.

   (a) Show that φ is a ring homomorphism.

   (b) Find a polynomial $p(x)$ such that $\ker(\varphi) = \langle p(x) \rangle$, and justify your answer.

   (c) Show that $\varphi(f(x)) = \varphi(g(x))$ if and only if $f(x) - g(x) \in \ker(\varphi)$.

   (d) Determine the image of φ.

   (e) Is $\ker(\varphi)$ a prime ideal? Is it maximal? Justify your answers.

   (f) What can you conclude using the First Isomorphism Theorem?

6. Let $G = \langle 2 \rangle$ and $H = \langle 8 \rangle$ be subsets of $\mathbb{Z}$.

   (a) Viewing G and H as additive groups, show that the quotient group G/H is isomorphic to $\mathbb{Z}/4\mathbb{Z}$.

   (b) Viewing G and H as rings, determine whether the quotient G/H is isomorphic to the ring $\mathbb{Z}/4\mathbb{Z}$. Justify your answer.

   (c) Now let $G = \langle n \rangle$ and $H = \langle m \rangle \subseteq \mathbb{Z}$ with $n \mid m$. State and justify a general result comparing the structure of G/H as a group and as a ring.

7. Let $R = (\mathbb{Z}/2\mathbb{Z})[x]/(x^2+1)$.

   (a) List all elements of R. How many elements does R have?

   (b) Is R an integral domain? Justify your answer.

   (c) Is R a field? Justify your answer.

8. Let $\mathbb{F}_3$ be the field with 3 elements.

   (a) Determine all monic irreducible polynomials of degree 2 over $\mathbb{F}_3$.

   (b) Construct the field $\mathbb{F}_9$ explicitly using one such polynomial.

   (c) Determine the multiplicative order of every nonzero element in $\mathbb{F}_9$.

Section 3: Galois Theory

9. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

   (a) Compute $[K : \mathbb{Q}]$.

   (b) Show that K is the splitting field of a polynomial over $\mathbb{Q}$.

   (c) Prove that $K/\mathbb{Q}$ is a normal extension.

   (d) Determine $\mathrm{Gal}(K/\mathbb{Q})$ and identify it as a known group.

   (e) Use the Galois correspondence to describe all intermediate fields of $K/\mathbb{Q}$ and determine which are normal over $\mathbb{Q}$.

10. Let $f(x) = x^3 + bx + c$ be an irreducible separable polynomial over a field K, and let S be its splitting field.

    (a) Show that $\mathrm{Gal}(S/K) \cong S_3$ if and only if the discriminant $D = -4b^3 - 27c^2$ is not a square in K.

    (b) Show that if D is a square in K, then $[S : K] = 3$.

11. Let $F = \mathbb{F}_{p^n}$ be a finite field.

    (a) Show that the map $\sigma(x) = x^p$ is an automorphism of F.

    (b) Prove that $\mathrm{Gal}(F/\mathbb{F}_p)$ is cyclic.

    (c) Determine the order of $\mathrm{Gal}(F/\mathbb{F}_p)$.

    (d) Describe all subfields of F in terms of the divisors of n.