SWARTHMORE COLLEGE
DEPARTMENT OF MATHEMATICS AND STATISTICS
2019 ALGEBRA HONORS EXAMINATION

Instructions: This exam contains nine problems. Try to solve **six** problems as completely a possible. The exam is divided into three sections; please attempt to solve at least one, ideally more than one, problem from each section. Once you are satisfied with your responses to six problems, make a second pass through the exam and complete as many parts of the remaining problems as possible. For problems with parts, you are allowed to assume the truth of earlier parts when working on a later part. I am interested in your thoughts on a problem and attempts at special cases even if you do not completely solve the problem. Please justify your reasoning as fully as possible.

Section I.

1. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$. For each $i \in \Omega$ the set $\{\, \sigma \in S_6 \mid \sigma(i) = i \,\}$ is a subgroup of the symmetric group S_6 that is isomorphic to S_5 and has exactly two orbits on Ω, namely $\{i\}$ and $\Omega - \{i\}$ (do not prove this). The purpose of this problem is to prove that S_6 contains a subgroup K that is isomorphic to S_5 and that has just one orbit on Ω.

(a) Prove that S_5 contains exactly 6 Sylow 5-subgroups.

(b) Prove that S_5 contains a subgroup N such that $|S_5 : N| = 6$.

(c) Prove that the rule $(N\alpha) \cdot \beta = N\alpha\beta$ defines a group action of S_5 on the set Δ consisting of the right cosets of N in S_5. Prove that S_5 has just one orbit on Δ.

(d) Prove that S_6 contains a subgroup K that is isomorphic to S_5 such that K has exactly one orbit in its action on Ω.

2. Let G be a group whose identity element is denoted 1. For each $x \in G$ the set $\mathbf{C}_G(x) = \{\, y \in G \mid xy = yx \,\}$ is a subgroup of G (do not prove this). We define a binary relation $\sim$ on the set $G^* = G - \{1\}$ as follows. For $x, y \in G^*$ we write $x \sim y$ iff $xy = yx$.

(a) Prove that the relation $\sim$ is reflexive and symmetric.

(b) Suppose G is nonabelian and that its center $\mathbf{Z}(G)$ contains more than one element. Prove that the relation $\sim$ is not transitive.

(c) Prove that $\sim$ is transitive if and only if $\mathbf{C}_G(x)$ is abelian for every $x \in G^*$.

(d) Let G be a finite group for which $\sim$ is an equivalence relation (i.e. for which $\sim$ is transitive) and let $\mathcal{C}_1, \ldots, \mathcal{C}_r$ denote the distinct equivalence classes. Prove that for each $i \in \{1, \ldots, r\}$ the set $\mathcal{C}_i \cup \{1\}$ is a maximal abelian subgroup of G (i.e. an abelian subgroup that is not contained inside any larger abelian subgroup of G).

(e) Find an example of a finite group for which $\sim$ is an equivalence relation.

3. Let G be a group acting on a finite set Ω with $|\Omega| > 1$. Suppose this action is transitive, which means that for each pair α, $\beta \in \Omega$ there exists $g \in G$ such that $\alpha \cdot g = \beta$. Now let G act on the Cartesian product $\Omega \times \Omega$ by $(\alpha, \beta) \cdot g = (\alpha \cdot g, \beta \cdot g)$. Fix an arbitrary point $\delta \in \Omega$ and consider the stabilizer subgroup $G_\delta = \{\, g \in G \,|\, \delta \cdot g = \delta \,\}$. Let n denote the number of orbits in the action of G_δ on Ω.

(a) Let $\delta_1, \ldots, \delta_n$ be representatives for the distinct orbits in the action of G_δ on Ω. (We may assume $\delta_1 = \delta$.) For $i \in \{1, \ldots, n\}$ let $\mathcal{O}_i$ denote the orbit that contains (δ, δ_i) in the action of G on $\Omega \times \Omega$. Prove that the sets $\mathcal{O}_i$ and $\mathcal{O}_j$ are distinct whenever $i \neq j$.

(b) Prove that G has exactly n orbits in its action on $\Omega \times \Omega$.

(c) Suppose $n = 2$. Prove there does not exist any subgroup H such that $G_\delta < H < G$.

Section II.

4. The ring of Gaussian integers $\mathbb{Z}[i] = \{\, a + bi \,|\, a, b \in \mathbb{Z} \,\}$ is a subring of the complex numbers $\mathbb{C}$ (do not prove this). Fix integers m, n such that $n > 0$ and let I denote the principal ideal of $\mathbb{Z}[i]$ generated by $m + ni$. Let R denote the quotient ring $\mathbb{Z}[i]/I$.

(a) Prove that every element of R can be written in the form $k + ri + I$ for some pair of integers k, r such that $0 \leq r < n$.

(b) Prove that $m^2 + n^2 + I = 0 + I$.

(c) Prove that R is finite.

(d) Take $m = -2$ and $n = 1$. Prove that the element $1 + I$ in R has additive order 5 and then conclude that R is isomorphic to $\mathbb{Z}/5\mathbb{Z}$. (Be sure to provide sufficient detail.)

5. Let F be a field. Let R be a subring of F that contains the unity element of F. Suppose that M is a maximal ideal of R. We define the sets

$$S = \left\{ \frac{a}{b} \mid a \in R,\ b \in R - M \right\} \quad \text{and} \quad J = \left\{ \frac{a}{b} \mid a \in M,\ b \in R - M \right\}.$$

(a) Prove that S is a subring of F and that S contains R.

(b) Prove that J is an ideal of S.

(c) Prove that S contains the multiplicative inverse of every element in $S - J$.

(d) Prove that every proper ideal of S is contained in J.

6. Let A be an ideal of a commutative ring R. The **nil radical** of A is the set $N(A)$ consisting of all those elements $r \in R$ such that $r^n \in A$ for some positive integer n that depends on the element r.

(a) Prove that $N(A)$ is an ideal of R.

(b) For the ring $R = \mathbb{Z}/100\mathbb{Z} = \{0, 1, \ldots, 99\}$, describe or list all the elements of each of the following ideals: $N(\langle 0 \rangle)$, $N(\langle 2 \rangle)$, $N(\langle 4 \rangle)$, $N(\langle 10 \rangle)$.

(c) An element x in a ring is said to be **nilpotent** in case $x^n = 0$ for some positive integer n. For an arbitrary commutative ring R, prove that the quotient ring $R/N(\langle 0 \rangle)$ does not contain any nonzero nilpotent elements.

Section III.

7. Let E denote the field $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}, \sqrt[5]{2}, \ldots)$ where $\mathbb{Q}$ is the field of rational numbers.

(a) Prove that E is not a finite extension of $\mathbb{Q}$.

(b) Prove that E is an algebraic extension of $\mathbb{Q}$. (It might be useful to argue that E is equal to the union of an infinite sequence of fields, each contained in the next.)

(c) Prove that $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt[3]{2}, \sqrt[4]{3})$ is a field extension of degree 12.

8. Let $2^{1/4}$ be the positive real 4th root of 2. Let E denote the subfield $\mathbb{Q}(i, 2^{1/4})$ of $\mathbb{C}$.

(a) Write down all the zeros of the polynomial $x^4 - 2$ in $\mathbb{C}$.

(b) Prove that $x^4 - 2$ is irreducible over $\mathbb{Q}(i)$.

(c) Write down a basis for the vector space E over $\mathbb{Q}(i)$.

(d) What are the possibilities for $\sigma(2^{1/4})$ if $\sigma \in \mathrm{Gal}(E/\mathbb{Q}(i))$?

(e) Suppose $\alpha \in \mathrm{Gal}(E/\mathbb{Q}(i))$ satisfies the condition $\alpha(2^{1/4}) = i2^{1/4}$. Consider the subgroup $H = \langle \alpha^2 \rangle$ of $\mathrm{Gal}(E/\mathbb{Q}(i))$. Determine the fixed field E_H.

9. Let $\omega = e^{2\pi i/6} \in \mathbb{C}$. We define the matrices

$$a = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We define the subgroups $A = \langle a \rangle$, $B = \langle b \rangle$, $G = \langle a, b \rangle$ of the general linear group $GL(2, \mathbb{C})$.

(a) Prove that $b^{-1}a^i b = a^{-i}$ for every positive integer i. Use this to argue that $A \triangleleft G$.

(b) Compute each of $|A|$, $|B|$, $|A \cap B|$, and $|G|$.

(c) Prove that $G = A \cup bA$ is a disjoint union. (Thus each element of G may be written in the "standard form" $b^i a^j$ where $i \in \{0, 1\}$ and j is an appropriate nonnegative integer.)

(d) List the elements of the commutator subgroup G'. Is G/G' cyclic? (By definition G' is the subgroup generated by all the elements of the form $x^{-1}y^{-1}xy$ where x, $y \in G$.)

(e) Determine the distinct conjugacy classes of elements of G.

(f) Compute the character table of G.