Swarthmore College
Department of Mathematics and Statistics
Honors Examination: Algebra-C

Spring 2003

Instructions: This exam that contains 9 problems. Try to solve *six* problems as completely as possible. Beyond that, turn in any solutions or partial solutions that you can get done.

Advice: I am interested in your thoughts on the problem even if they do not completely solve it. In particular, turn in your solution even if you can't do all the parts of a multiple part problem. You might also formulate and solve special cases that you can think of. Where there are multiple parts to a problem, you might be able to answer a later part without solving all the earlier ones.

1. Let $\phi : G \to H$ be a group homomorphism. Let $x, y \in H$ be in the image of ϕ. Find a bijection between the sets $\phi^{-1}(x)$ and $\phi^{-1}(y)$. Justify your answer.

2. (a) Let σ be an automorphism of a group G (an isomorphism from G onto G). Prove that σ permutes the conjugacy classes of G. That is if $\mathcal{K}$ is a conjugacy class of G, then $\sigma(\mathcal{K})$ is a conjugacy class of G. (*Recall that $\mathcal{K}$ is a conjugacy class means that for some $h \in G$, $\mathcal{K} = \{g^{-1}hg | g \in G\}$.*)

   (b) Prove that any automorphism of the symmetric group S_5 sends transpositions to transpositions. Hint: think about the sizes of the conjugacy classes of elements of order two.

3. Let G be a finite group. Cayley's theorem says that G is isomorphic to a subgroup of Perm(G), where Perm(G) is the group of all permutations of G. Let $\pi : G \to \text{Perm}(G)$ be the homomorphism from G into Perm(G).

   (a) Let $|G| = n$ and let $x \in G$ with $|x| = m$. Describe the cycle structure of the permutation $\pi(x)$. (Here $|x|$ denotes the order of the element x).

   (b) Prove that $\pi(x)$ is an odd permutation if and only if $|x|$ is even and $|G|/|x|$ is odd.

   (c) Prove that if G contains an element x with $|x|$ even and $|G|/|x|$ odd, then G has a subgroup of index 2 and, thus, G is not simple.

4. Let $a = \sqrt{2}\omega \in \mathbb{C}$, where $\omega = e^{2\pi i/3}$.

   (a) Find the minimal polynomial for a over $\mathbb{Q}$.

   (b) Find a basis for $\mathbb{Q}(a)$ over $\mathbb{Q}$ (justify your answer).

5. Let φ denote the Euler φ function.

   (a) Prove: $\sum_{d|n} \varphi(d) = n$. (Consider the orders of elements in $\mathbb{Z}_n$).

   (b) Let p be a prime and find an element of order n in the multiplicative group $\mathbb{Z}^*_{p^n-1}$. Prove that $n|\varphi(p^n - 1)$.

   (c) Prove Euler's Theorem: if $(a, n) = 1$ then $a^{\varphi(n)} \equiv 1 \text{mod} n$.

   (d) Determine the last two digits of 3^{400}.

6. (a) Find an ideal I of $\mathbb{Z}_8[x]$ such that $\mathbb{Z}_8[x]/I$ is a field.
   (b) Find an ideal I of $\mathbb{Z}_8[x]$ such that $\mathbb{Z}_8[x]/I$ is an intgral domain but not a field.

7. Let R be a commutative ring with unity 1, and let $G = \{g_1, g_2, \ldots, g_n\}$ be a finite group. Define $R[G]$ as

$$R[G] = \left\{ \sum_{i=1}^{n} x_i g_i \;\middle|\; x_i \in R \right\}$$

with addition and multiplication given (like polynomials) by

$$\left(\sum_{i=1}^{n} x_i g_i\right) + \left(\sum_{i=1}^{n} y_i g_i\right) = \sum_{i=1}^{n} (x_i + y_i) g_i.$$

$$\left(\sum_{i=1}^{n} x_i g_i\right) \cdot \left(\sum_{i=1}^{n} y_i g_i\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j (g_i g_j).$$

You do not have to check the properties, but $R[G]$ forms a ring (called a group ring).

   (a) Show that $R[G]$ contains an isomorphic copy of G and an isomorphic copy of R.
   (b) The center of a ring A is the set $\{z \in A \mid za = az \text{ for all } a \in A\}$. Show that $\bar{g} = g_1 + g_2 + \cdots + g_n$ is in the center of $R[G]$
   (c) Define $\phi : R[G] \to R$ by $\phi\left(\sum_{i=1}^{n} x_i g_i\right) = \sum_{i=1}^{n} x_i$. Show that ϕ is a ring homomorphism.
   (d) Describe the factor ring $R[G]/\ker(\phi)$.

8. Let $\gamma = (1 + \sqrt{5})/2$ (the golden ratio).
   (a) Find an infinite continued fraction for γ.
   (b) Hurwitz proved that for any irrational number ξ, there are always infinitely many convergents h_n/k_n to ξ such that

$$\left|\xi - \frac{h_n}{k_n}\right| < \frac{1}{k_n^2\sqrt{5}}$$

   Find (with proof) an infinite set of convergents h_n/k_n that satisfy the inequality for the golden ratio γ.
   (c) Show that $\sqrt{5}$ in the previous inequality is the best possible in the sense that if $c > \sqrt{5}$, then there are at most finitely many h_n/k_n for which

$$\left|\xi - \frac{h_n}{k_n}\right| < \frac{1}{k_n^2 c}.$$

9. Let D be an integer that is not a perfect square in $\mathbb{Z}$. Define the following subring of $\mathbb{C}$,

$$\mathbb{Z}[\sqrt{D}] = \left\{ a + b\sqrt{D} \;\middle|\; a, b \in \mathbb{Z} \right\}.$$

Define $N : \mathbb{Z}[\sqrt{D}] \to \mathbb{Z}$ by $N(a + b\sqrt{D}) = a^2 - Db^2$.

   (a) Show that $N(xy) = N(x)N(y)$.
   (b) Show that u is a unit in $\mathbb{Z}[\sqrt{D}]$ if and only if $N(u) = \pm 1$ (Hint: for one direction find $u^{-1} \in \mathbb{C}$).
   (c) Find all the units in $\mathbb{Z}[\sqrt{-1}]$.
   (d) Show that $\mathbb{Z}[\sqrt{2}]$ has an infinite number of units (Hint: if u is a unit, consider u^n).