Swarthmore College
Department of Mathematics and Statistics
Honors Examination: Algebra-B

Spring 2003

Instructions: This exam contains 9 problems. Try to solve *six* problems as completely as possible. *Beyond that, turn in any solutions or partial solutions that you can get done.*

Advice: I am interested in your thoughts on the problem even if they do not completely solve it. In particular, turn in your solution even if you can't do all the parts of a multiple part problem. You might also formulate and solve special cases that you can think of. Where there are multiple parts to a problem, you might be able to answer a later part without solving all the earlier ones.

1. Let $\phi : G \to H$ be a group homomorphism. Let $x, y \in H$ be in the image of ϕ. Find a bijection between the sets $\phi^{-1}(x)$ and $\phi^{-1}(y)$. Justify your answer.

2. (a) Let σ be an automorphism of a group G (an isomorphism from G onto G). Prove that σ permutes the conjugacy classes of G. That is if $\mathcal{K}$ is a conjugacy class of G, then $\sigma(\mathcal{K})$ is a conjugacy class of G. (*Recall that $\mathcal{K}$ is a conjugacy class means that for some $h \in G$, $\mathcal{K} = \{g^{-1}hg | g \in G\}$.*)

   (b) Prove that any automorphism of the symmetric group S_5 sends transpositions to transpositions. Hint: *think about the sizes of the conjugacy classes of elements of order two.*

3. Let G be a finite group. Cayley's theorem says that G is isomorphic to a subgroup of Perm(G), where Perm(G) is the group of all permutations of G. Let $\pi : G \to \text{Perm}(G)$ be the homomorphism from G into Perm(G).

   (a) Let $|G| = n$ and let $x \in G$ with $|x| = m$. Describe the cycle structure of the permutation $\pi(x)$. (Here $|x|$ denotes the order of the element x).

   (b) Prove that $\pi(x)$ is an odd permutation if and only if $|x|$ is even and $|G|/|x|$ is odd.

   (c) Prove that if G contains an element x with $|x|$ even and $|G|/|x|$ odd, then G has a subgroup of index 2 and, thus, G is not simple.

4. Let $a = \sqrt{2}\omega \in \mathbb{C}$, where $\omega = e^{2\pi i/3}$.

   (a) *Find the minimal polynomial for a over $\mathbb{Q}$.*

   (b) Find a basis for $\mathbb{Q}(a)$ over $\mathbb{Q}$ (justify your answer).

5. Let $F \subseteq E$, where E is a field extension of F and $|F| = q < \infty$. Show that $F = \{\alpha \in E \mid \alpha^q = \alpha\}$. Hint: To show "$\subseteq$," use the fact that $F \setminus \{0\}$ is a finite group, and to show "$\supseteq$," count.

6. Let G be a group. Two subgroups H and J of G are *conjugate* in G if $gHg^{-1} = J$ for some $g \in G$. Let $F \subseteq E$ be a finite field extension. Two intermediate fields $F \subseteq K \subseteq E$ and $F \subseteq L \subseteq E$ are *F-isomorphic* if $\alpha(K) = L$ for some α in the Galois group $G(E/F)$.

   Let $F \subseteq E$ be a finite Galois extension. Let $F \subseteq K \subseteq E$ and $F \subseteq L \subseteq E$ be intermediate fields. Prove that K and L are F-isomorphic if and only if $G(E/K)$ and $G(E/L)$ are conjugate in $G(E/F)$.

7. Let R be a commutative ring with unity 1, and let $G = \{g_1, g_2, \ldots, g_n\}$ be a finite group. Define $R[G]$ as

$$R[G] = \left\{ \sum_{i=1}^{n} x_i g_i \;\middle|\; x_i \in R \right\}$$

with addition and multiplication given (like polynomials) by

$$\left(\sum_{i=1}^{n} x_i g_i\right) + \left(\sum_{i=1}^{n} y_i g_i\right) = \sum_{i=1}^{n} (x_i + y_i) g_i.$$

$$\left(\sum_{i=1}^{n} x_i g_i\right) \cdot \left(\sum_{i=1}^{n} y_i g_i\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j (g_i g_j).$$

You do not have to check the properties, but $R[G]$ forms a ring (called a group ring).

(a) Show that $R[G]$ contains an isomorphic copy of G and an isomorphic copy of R.

(b) The center of a ring A is the set $\{z \in A \mid za = az \text{ for all } a \in A\}$. Show that $\bar{g} = g_1 + g_2 + \cdots + g_n$ is in the center of $R[G]$

(c) Define $\phi : R[G] \to R$ by $\phi\left(\sum_{i=1}^{n} x_i g_i\right) = \sum_{i=1}^{n} x_i$. Show that ϕ is a ring homomorphism.

(d) Describe the factor ring $R[G]/\ker(\phi)$.

8. Let $V = \mathbb{C}^n$ with basis $v_1, \ldots, v_n$ be the permutation representation of the symmetric group S_n. That is $\sigma(v_i) = v_{\sigma(i)}$ for $\sigma \in S_n$. Let the subspace W be defined by

$$W = \left\{ \sum_{i=1}^{n} \alpha_i v_i \;\middle|\; \alpha_1 + \alpha_2 + \cdots + \alpha_n = 0 \right\}$$

(a) Show that W is an S_n-sub*module.*

(b) Determine how to compute the character $\chi_W(\sigma)$ simply in terms of the permutation σ.

(c) Now view $S_{n-1} \subseteq S_n$ as the permutations which fix n. Then the elements of S_{n-1} act on W (since they are in S_n and S_n acts on W). This is the restriction of the representation W from S_n to S_{n-1} (you do not need to prove that it is a representation). Show that W is the permutation representation of S_{n-1}.

9. Suppose that K is a normal subgroup of a finite group G and that S is a Sylow p-subgroup of G. Prove that $K \cap S$ is a Sylow p-subgroup of K.