Swarthmore College
Department of Mathematics and Statistics
Honors Examination
Algebra

Spring 2004

Instructions: This exam contains 10 problems. Try to solve about *six* problems as completely as possible. Beyond that, turn in any solutions or partial solutions that you can get done. I am interested in your thoughts on the problem even if they do not completely solve it. In particular, turn in your solution even if you can't do all the parts of a multiple part problem. You might also formulate and solve special cases that you can think of. Where there are multiple parts to a problem, you can answer a later part without solving all the earlier ones. The problems are not necessarily in increasing order of difficulty.

1. Let A and B be subgroups of a group G with $A \cap B = \{1\}$.

   (a) Show that if A and B are normal subgroups, then $ab = ba$ for all $a \in A$ and $b \in B$. (Hint: think about $b^{-1}aba^{-1}$).

   (b) Show that if $ab = ba$ for all $a \in A$ and $b \in B$, then $AB = \{ab \mid a \in A, b \in B\}$ is a subgroup of G and $AB \cong A \times B$.

2. Exhibit all the permutations in S_7 that commute with $\alpha = (1\ 2)(3\ 4\ 5)$. Justify your answer.

3. An automorphism of a group G is an isomorphism from G to itself. A subgroup H of G is called characteristic, denoted $H\,\mathbf{char}\,G$, if every automorphism of G maps H to itself (that is, $\phi(H) = H$ for all automorphisms ϕ of G).

   (a) Prove that characteristic subgroups are normal.

   (b) Prove that: if $K\,\mathbf{char}\,H$ and $H \triangleleft G$, then $K \triangleleft G$ (here $\triangleleft$ denotes normal subgroup).

   (c) Give an example of a normal subgroup that is not characteristic.

4. A group is simple if it has no proper nontrivial normal subgroups. This problem will prove that there is no simple group G of order 45. By way of contradiction, suppose that G is a simple group of order 45 (keep in mind that somewhere below you need to use the assumption that G is simple).

   (a) The Sylow theorems tell us that G has a 3-subgroup P of order 9. There are 5 cosets of P. Let's call them $\{g_1P, g_2P, g_3P, g_4P, g_5P\}$. Show that left multiplication by an element $g \in G$ gives a permutation of these cosets.

   (b) Part (a) allows us to associate each element of G with a permutation in S_5. Thus, it gives us a map $\phi : G \to S_5$. Show that the map ϕ is injective (i.e., one-to-one).

   (c) Argue that S_5 does not have a subgroup of order 45, and thus G cannot be simple.

5. True/False? Justify Your answers

   (a) $4x^2 + 6x + 3$ is a unit in $\mathbb{Z}_8[x]$.

   (b) $\mathbb{Z}_7[\sqrt{3}]$ is a field.

   (c) (x, y) is a maximal ideal in $\mathbb{Z}[x, y]$

   (*Notation:* $\mathbb{Z}[x, y]$ *is the ring of polynomials in two variables* x *and* y *with integer coefficients, and* (x, y) *is the ideal generated by* x *and* y).

6. Let R be a commutative ring with 1 and let $s \in R$. Define the *annhilator* of s to be $Ann(s) = \{a \in R \mid sa = 0\}$.

   (a) Prove that $Ann(s)$ is an ideal of R.

   (b) Describe $Ann(s)$ when s is a unit.

   (c) If $e \in R$ satisfies $e^2 = e$, then show that $Ann(e) = (1 - e)R$.

   (d) It is tempting to think that $s + Ann(s)$ is not a zero divisor in the quotient ring $R/Ann(s)$ (since we are dividing out all the stuff that sends s to 0). Find a counter-example to this statement in $\mathbb{Z}_{12}$.

7. Let R be a commutative ring with 1 and let $a \in R$. Let $R' = R[x]/(ax - 1)$.

   (a) Describe R' in the case where a is a unit.

   (b) Describe R' in the case where a is nilpotent, i.e., $a^n = 0$ for $n \in \mathbb{Z}^{>0}$.

8. Let $f(x), g(x) \in \mathbb{Q}[x]$ be irreducible polynomials with a common zero $z \in \mathbb{C}$. Prove that they generate the same principal ideal $(f) = (g)$. (Hint: think about the ideal (f, g) generated by them both).

9. Let F be a finite field with q elements and let a be a nonzero element of F. Prove that if n divides $q - 1$, then $x^n - a$ has either no solutions in F or has n distinct solutions in F. (Hint: consider the multiplicative group $F^\times = F \setminus \{0\}$).

10. Let $\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2\}$ be a field of order 4. Let G be the group of invertible 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with entries $a, b, c, d \in \mathbb{F}_4$, whose column sums are 1 (i.e., $a + c = b + d = 1$; these are called stochastic matrices).

    (a) Give the multiplication and addition table for $\mathbb{F}_4$.

    (b) Show that G is a nonabelian group of order 12.

    (c) Up to isomorphism, there are 3 nonabelian groups of order 12: the dihedral group D_6, the alternating group A_4, and $Q_6 = \langle\ s, t \mid s^6 = 1, s^3 = t^2, sts = t\ \rangle$. Which group is it? (you do not need to exhibit an isomorphism).