ALGEBRA EXAM

SPRING 2021

This exam consists of 9 problems. You are not expected to solve them all, but rather should work on the ones you find interesting and approachable.

I am interested in seeing how you approach the various problems, so please turn in your solutions to a problem even if you can only make progress on some of the individual parts. If you find it useful, you may assume an earlier part of a problem when working on the later parts even if you haven't been able to solve it. However, please bear in mind that I would rather see substantial progress on a few problems than a handful of computations for every problem.

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(1) Let $G = \mathbb{Z}/24\mathbb{Z}$, let $H = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$, and let $J = \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. View G, H, and J as groups with the operation $+$.
   (a) Find a finite field F such that the group of units F^* of F is isomorphic to G. Explain your reasoning.
   (b) Find a primitive element of F^*. Use this element to describe an isomorphism between F^* and G.
   (c) Can you construct a finite field with group of units isomorphic to H? Why or why not?
   (d) Can you construct a finite field with group of units isomorphic to J? Why or why not?

(2) Let $GL(3, \mathbb{R})$ be the group of invertible 3×3 real matrices and let T be the tetrahedron with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(-1, -1, -1)$.
   (a) Let G be the subgroup of $GL(3, \mathbb{R})$ that maps T to itself. Show that elements of G are matrices with determinant ± 1. (Hint: vertices must map to vertices.)
   (b) Show that G is a finite group. Can you describe G using an isomorphism to a well-known group? What is its order?
   (c) Let H be the subgroup of G consisting of matrices with determinant 1. We call H the *rotational symmetries* of T. Can you describe H using an isomorphism to a well-known group? What is its order?
   (d) What is the order of the smallest nontrivial subgroup of H? What does this observation tell us about rotations of the tetrahedron?

(3) Let p be a prime number. Let G be the group of 3×3 upper triangular matrices over $\mathbb{Z}/p\mathbb{Z}$ with 1s on the diagonal. That is,

$$G = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}.$$

   (a) Show that G is a non-abelian group of order p^3.
   (b) Show that if $p \neq 2$, then x^p is the identity matrix for all $x \in G$.

(c) There are two non-abelian groups of order 8, the dihedral group and the quaternion group. When $p = 2$, which of these groups is G isomorphic to? Explain your answer.

(4) The smallest nonabelian simple group is A_5, the alternating group of degree 5. Let G be the second-smallest nonabelian simple group. This group has 168 elements. (It may be constructed as a quotient of $SL(2,7)$, the group of 2×2 matrices with entries in $\mathbb{Z}/7\mathbb{Z}$ and determinant 1.)
  (a) Show that G has at least 6 elements of order 7.
  (b) How many elements of order 7 does G have?

(5) Let H and K be subgroups of a finite group G and let $H \times K$ act on G by $(h,k) \cdot x = hxk^{-1}$ for all $h \in H$, $k \in K$, and $x \in G$.
  (a) Show that the mapping described above is a group action.
  (b) Show that the orbit of $x \in G$ is the *double coset*

$$HxK = \{hxk \mid h \in H, k \in K\}.$$

  (c) Show that the stabilizer $S(x)$ satisfies

$$|S(x)| = |H \cap xKx^{-1}| = |x^{-1}Hx \cap K|.$$

  (d) Prove *Frobenius' Theorem*: If Hx_1K, Hx_2K, ..., Hx_nK are the distinct double cosets of G, then

$$|G| = \sum_{i=1}^{n} \frac{|H||K|}{x_i^{-1}Hx_i \cap K}.$$

(6) Consider the *real trigonometric polynomials* of the form

$$a_0 + \sum_{n=1}^{k} (a_n \cos(nt) + b_n \sin(nt)),$$

where $a_0, \ldots, a_k$ and $b_1, \ldots, b_k$ are real numbers. The *degree* of a nonzero trigonometric polynomial is the largest integer n such that a_n and b_n are not both zero. Using standard trigonometric identities, one may prove the following fact:

The product of a trigonometric polynomial of degree m and a trigonometric polynomial of degree n is a trigonometric polynomial of degree $m + n$.

(You do not need to prove this fact.)
  (a) Explain why the trigonometric polynomials form a commutative ring. What is the additive identity? What is the multiplicative identity?
  (b) Show that the trigonometric polynomials are an integral domain. What are the units?
  (c) Show that trigonometric polynomials of degree 1 are irreducible.
  (d) Show that the trigonometric polynomials are not a unique factorization domain. (Hint: you may use the familiar identity $\sin^2 t = 1 - \cos^2 t$.)

(7) Let p be a prime, let $f(x) = x^p - x - 1$, and let u be a root of $f(x)$ in a splitting field E of $f(x)$ over $\mathbb{Z}_p$.
  (a) Show that $u + a$ is a root of $f(x)$ for any $a \in \mathbb{Z}_p$.
  (b) Show that $E = \mathbb{Z}_p(u)$.

(c) Show that $f(x)$ is irreducible over $\mathbb{Z}_p$. (Hint: suppose $p(x)$ is a factor of $f(x)$ of degree d. What do you know about the coefficient of x^{d-1} in $p(x)$?)
(d) What is the order of E?

(8) Let E be the splitting field of the polynomial $x^3 + 7$ over $\mathbb{Q}$. Let G be the Galois group $\mathrm{Gal}(E : \mathbb{Q})$.
(a) What group is G? (Hint: you may want to use the fact that $(\frac{1}{2} - i\frac{\sqrt{3}}{2})^3 = -1$.)
(b) Show that E is a Galois extension of $\mathbb{Q}$.
(c) Find the lattice of subgroups of G.
(d) Describe the corresponding lattice of subfields. (Your explanation should allow me to identify what elements are in each subfield.)
(e) Find a nontrivial normal subgroup H of G, and let K be the corresponding field. Compute $\mathrm{Gal}(E : K)$ and $\mathrm{Gal}(K : F)$.

(9) Let G be a finite group, let V be a finitely generated vector space, and let $\phi : G \to GL(V)$ be a representation of G. The *centralizer* of ϕ is defined to be the set of all linear transformations $A : V \to V$ such that $A\phi(G) = \phi(g)A$ for all $g \in G$ (i.e., the linear transformations of V which commute with all the $\phi(g)$.)
(a) Prove that a linear transformation A from V to V is in the centralizer of ϕ if and only if it is an FG-module homomorphism from V to itself.
(b) Explain why the centralizer of ϕ is a ring.
(c) Show that if z is in the center of G then $\phi(z)$ is in the centralizer of ϕ.
(d) Let C_4 be the cyclic group with four elements and let $G = C_4 \times C_4$. Construct two distinct representations of G and describe the centralizer in each case.