Swarthmore College
Department of Mathematics and Statistics
Honors Examination: Algebra
Spring 2015

Instructions: This exam contains nine problems. Try to solve six problems as completely as possible. Do not be concerned if some portions of the exam are unfamiliar; you have a number of choices so that you do not need to answer every question. Once you are satisfied with your responses to six problems, make a second pass through the exam and complete as many parts of the remaining problems as possible. I am interested in your thoughts on a problem even if you do not completely solve it. In particular, submit your solution even if you cannot do all the parts of a problem. When there are multiple parts, you are permitted to address a later part without solving all the earlier ones.

1. Let $\mathbb{Q}$ be the additive group of rational numbers and $\mathbb{Z}$ its subgroup of integers.

(a) If n is a positive integer, show that $\mathbb{Q}/\mathbb{Z}$ has an element of order n.

(b) If n is a positive integer, show that $\mathbb{Q}/\mathbb{Z}$ has a unique subgroup of order n.

(c) Show that every finite subgroup of $\mathbb{Q}/\mathbb{Z}$ is cyclic.

(d) Does $\mathbb{Q}/\mathbb{Z}$ have a proper infinite subgroup that is cyclic? If so, provide an example. Otherwise, show that no such subgroup exists.

(e) Does $\mathbb{Q}/\mathbb{Z}$ have a proper infinite subgroup that is not cyclic? If so, provide an example. Otherwise, show that no such subgroup exists.

2. For a prime integer p, denote the finite field with p elements by $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$. Addition and multiplication are defined modulo p. Note that $\mathbb{F}_p \cong \mathbb{Z}/(p)$.

(a) Determine all monic polynomials $f(X) \in \mathbb{F}_3[X]$ of degree 2 that are irreducible over $\mathbb{F}_3$.

(b) Define $\mathbb{F}_9$ to be the quotient $\mathbb{F}_3[X]/\langle q(X)\rangle$ where $q(X) = X^2 + X + 2$. Note that there is a natural inclusion $\mathbb{F}_3 \hookrightarrow \mathbb{F}_9$. (You are not being asked to show this.) Let $\alpha \in \mathbb{F}_9$ be a root of the polynomial $q(X)$. Prove that $\{1, \alpha\}$ forms a basis for $\mathbb{F}_9$ as a vector space over $\mathbb{F}_3$.

(c) Write α^5 as a $\mathbb{F}_3$-linear combination of 1 and α.

(d) Find $\mathrm{irr}(\alpha^5, \mathbb{F}_3)$, which is defined as the monic polynomial in $\mathbb{F}_3[X]$ of smallest degree that has α^5 as a root.

3. Let p be a prime integer, and let ζ be a primitive p-th root of unity. Let α be a root of $X^p - 2$, and let K be the splitting field of $X^p - 2$ over $\mathbb{Q}$.

(a) Show that $X^p - 2$ is irreducible over $\mathbb{Q}$.

(b) Show that if $f(X) = 1 + X + \cdots + X^{p-2} + X^{p-1}$, then $f(X+1)$ is irreducible over $\mathbb{Q}$. Why must $f(X)$ also be irreducible over $\mathbb{Q}$?

(c) Let $\varphi \in \mathrm{Gal(K/\mathbb{Q})}$. Prove that $\varphi(\alpha) = \zeta^{c(\varphi)}\alpha$ and $\varphi(\zeta) = \zeta^{d(\varphi)}$ for some $c(\varphi), d(\varphi) \in \mathbb{F}_p$ with $d(\varphi) \neq 0$.

(d) Show that the function $\rho : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_p)$ given by

$$\rho(\varphi) = \begin{pmatrix} 1 & 0 \\ c(\varphi) & d(\varphi) \end{pmatrix}$$

is a homomorphism of groups whose image consists of all matrices of the form $\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}$.

4. For any positive integer n, we represent the group of permutations on $\{1, \ldots, n\}$ by S_n. For any set S, we represent its cardinality by $\#S$.

(a) Determine $\#\{\sigma \in S_8 \mid \sigma^{15} = \mathrm{id}\}$.

(b) Determine $\#\{\sigma \in S_8 \mid \sigma^2 = (12345)(678)\}$.

(c) Define $\rho = (12345)(678) \in S_8$ and let $\mathrm{Cl}(\rho)$ be the conjugacy class of ρ in S_8. Determine the cardinality of $\mathrm{Cl}(\rho)$, the cardinality of its centralizer, and the cardinality of its normalizer; that is, determine $\#\mathrm{Cl}(\rho)$, $\#C_{S_8}(\mathrm{Cl}(\rho))$, and $\#N_{S_8}(\mathrm{Cl}(\rho))$.

(d) For any positive integer n, there is a natural inclusion $S_n \hookrightarrow S_{n+1}$, where we identify S_n with the subgroup of S_{n+1} consisting of all permutations that map the index "$n+1$" to itself. (You are not being asked to prove this obvious statement.) Let $\sigma \in S_{n+1}$ such that $\sigma \notin S_n$. Show that $S_{n+1} = S_n \sigma S_n$.

(e) Prove that S_n is a maximal subgroup of S_{n+1} (when identified as in part (d) above).

5. Consider the ring $R = \mathbb{Z}[\sqrt{-7}]$.

(a) What are the units of R?

(b) Show that 2 is irreducible in R.

(c) Show that $1 + \sqrt{-7}$ is irreducible in R.

(d) Is R a unique factorization domain?

(e) Given a field K that is algebraic of finite degree over $\mathbb{Q}$, recall that the ring of integers of K is the collection of elements of K that are roots of monic polynomials in $\mathbb{Z}[X]$. What is the ring of integers of $\mathbb{Q}(\sqrt{-7})$?

6. Let G be a group, N be a normal subgroup of G, and Z be the center of G.

(a) If G/Z is cyclic, does it follow that G is abelian? Prove your claim.

(b) If G/Z is abelian, does it follow that G is abelian? Prove your claim.

(c) Prove that if G is solvable, then N and G/N are solvable.

(d) Prove that if N and G/N are solvable, then G is also solvable.

7. Let A_4 be the alternating group of order 12; that is, A_4 consists precisely of all even permutations in S_4.

(a) Demonstrate that $A_4 \cong \langle a, b \mid a^3 = b^3 = abab = 1 \rangle$.

(b) Using the matrices

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix},$$

construct a representation of A_4 on $\mathbb{R}^3$.

8. Let p be a prime integer.

(a) Let σ be an element of order p in $\mathrm{GL}_2(\mathbb{F}_\mathrm{p})$. Prove that σ is conjugate to a matrix of the form

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$

where $a \neq 0$.

(b) Determine the number of p-Sylow subgroups of $\mathrm{GL}_2(\mathbb{F}_\mathrm{p})$.

(c) What is the number of elements of order p in $\mathrm{GL}_2(\mathbb{F}_p)$?

9. Let R be a commutative ring with unity. Recall that an ideal P is prime if it is properly contained in R such that for all $xy \in P$, either $x \in P$ or $y \in P$.

(a) Let $P_1, \ldots, P_n$ be prime ideals and I be an ideal such that $I \subseteq \bigcup_{i=1}^n P_i$. Prove that $I \subseteq P_i$ for some index i.

(b) Let $I_1, \ldots, I_n$ be ideals and let P be a prime ideal such that $\bigcap_{i=1}^n I_i \subseteq P$. Prove that $I_i \subseteq P$ for some i.

(c) Prove that for any ideal I, the quotient R/I is an integral domain iff I is prime.