Swarthmore College
Department of Mathematics and Statistics
Honors Examination: Algebra

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Instructions: This exam consists of nine problems. Please try to do *six* of them as thoroughly as possible. Once you have done your best on those, make a second pass through the exam and do as many parts of the remaining problems as you can.

General hints and advice: If you get stuck, work out some examples or special cases. If it makes sense, formulate and solve an easier version of the problem. In general, I am interested in learning where your thoughts are going, even if you do not answer the question completely. In case you can relate a question to other material you have studied, not specifically addressed by the question, feel free to add additional commentary. When there are multiple parts to a problem, you may do them in any order. Please justify your reasoning as fully as possible.

1. (a) List as many non-isomorphic groups of order 12 as you can.
   (b) For each one, indicate the number of Sylow p-subgroups, for $p = 2$ and $p = 3$, and describe each Sylow p-subgroup.
   (c) For each one, describe the center and give its order.
   (d) For each group, give reasons why it is not isomorphic to any of the other groups you have listed.

2. (a) Prove that a group G of order 30 always has a proper normal subgroup.
   (b) Is it true that if G has order pqr, where p, q, and r are distinct primes, then G has a proper normal subgroup? If you need to assume additional conditions on p, q, and r, then state those conditions.

3. Let G be the group with presentation

$$\langle x, y, z \mid x^2y^2 = x^2z^2 = y^2z^2 = xyx^{-1}y^{-1} = xzx^{-1}z^{-1} = yzy^{-1}z^{-1} = 1\rangle.$$

   Compute the order of G and describe its structure.

4. Let $\mathbb{Z}[i]$ denote the Gaussian integers, and for $z \in \mathbb{Z}[i]$, let $\langle z\rangle$ denote the principal ideal generated by z. For each of the following z, describe the structure of the ring $R = \mathbb{Z}[i]/\langle z\rangle$.
   (a) $z = 1 + 2i$
   (b) $z = 1 + 3i$
   (c) $z = 5$

   You might want to answer questions such as: Is R a field? Is it an integral domain? What is $|R|$? Is R isomorphic to a direct sum of simpler rings?

5. Let $\alpha = \sqrt{3} + \sqrt{5} \in \mathbb{R}$. Let $f(x)$ denote the irreducible polynomial of α over $\mathbb{Q}$.
   (a) Compute $f(x)$ and explain why it is irreducible.
   (b) Show that $f(x)$ is reducible mod p for every prime p.

6. A set S of $n \times n$ complex matrices is *simultaneously diagonalizable over* $\mathbb{C}$ if there exists a single invertible complex matrix P such that $P^{-1}AP$ is diagonal, for every $A \in S$. Say whether the following statements are TRUE or FALSE, and justify your answer, e.g., by giving a brief argument or counterexample.

   (a) If A and B are diagonalizable complex matrices, then A and B are simultaneously diagonalizable.

   (b) If a finite set G of complex matrices forms a group, then every matrix in G is diagonalizable.

   (c) If a finite set G of complex matrices forms an abelian group, then G is simultaneously diagonalizable.

   (d) Any finite group of G of complex matrices is simultaneously diagonalizable.

7. For $n \in \mathbb{Z}$, let $\zeta_n = e^{2\pi i/n}$. For $\alpha \in \mathbb{C}$, let $\mathbb{Q}(\alpha)$ denote the smallest subfield of $\mathbb{C}$ containing α.

   (a) Is $\sqrt{3} \in \mathbb{Q}(\zeta_3)$?

   (b) Is $\zeta_6 \in \mathbb{Q}(\zeta_3)$?

   (c) Is $\zeta_3 \in \mathbb{Q}(\zeta_8)$?

   (d) Is $\zeta_8 \in \mathbb{Q}(\zeta_{12})$?

   Explain your reasoning fully in each case, stating any theorems that you are using.

8. If G is a permutation group acting on a finite set, let $\widehat{\rho}$ denote the representation which maps every element of G onto the corresponding permutation matrix. The character of $\widehat{\rho}$ is denoted $\widehat{\chi}$, and is called the *permutation character of* G.

   (a) What quantity associated with the permutation g is $\widehat{\chi}(g)$ computing?

   (b) Compute the character table of S_3, the symmetric group on three letters, and express its permutation character $\widehat{\chi}$ as a sum of irreducible characters.

   (c) Let $G = \mathbb{Z}_4$, regarded as a permutation group on $\{1, 2, 3, 4\}$ arranged cyclically. Compute the character table of $\mathbb{Z}_4$ and express its permutation character $\widehat{\chi}$ as a sum of irreducible characters.

   (d) Suppose that G has a single orbit. Prove that the permutation character $\widehat{\chi}$ contains the trivial character χ_0 with multiplicity 1.

9. Let $\alpha = \sqrt[3]{3} \in \mathbb{R}$ and $\omega = e^{2\pi i/3} \in \mathbb{C}$.

   (a) Compute the degrees of each of the following fields over the rationals:
   (i) $\mathbb{Q}(\alpha)$, (ii) $\mathbb{Q}(\omega)$, (iii) $\mathbb{Q}(\alpha\omega)$, (iv) $\mathbb{Q}(\alpha + \omega)$, (v) $\mathbb{Q}(\alpha, \omega)$, (vi) $\mathbb{Q}(\alpha + \alpha^2)$.

   (b) Are any of the fields in (a) equal to each other? Isomorphic to each other but unequal?

   (c) Which of the fields in (a) are splitting fields over $\mathbb{Q}$?

   (d) Let G denote the Galois group of $\mathbb{Q}(\alpha, \omega)$ over $\mathbb{Q}$. Make sketches showing all of the intermediate fields $\mathbb{Q} \subseteq K \subseteq \mathbb{Q}(\alpha, \omega)$ and the corresponding subgroups of G. Indicate which of the subgroups are normal in G, and state any theorems you are using to determine this.