Swarthmore College

Department of Mathematics and Statistics

2012 Honors Examination in Algebra

This exam contains 10 problems. Try to solve about *six* problems as completely as possible. Beyond that, turn in any solutions or partial solutions that you can get done. The problems are categorized into 3 parts; try to solve at least one, and ideally two, problems from each part.

I am interested in your thoughts on the problem even if they do not completely solve it. Turn in your solution even if you can't do all the parts of a multiple part problem. Where there are multiple parts to a problem, you can answer a later part without solving all the earlier ones.

Part A

1. Prove that the number of inner automorphisms of a group G equals the index of the center $[\mathsf{G} : Z(\mathsf{G})]$.

2. Let $\mathbb{Z}[i] = \{ a + bi \mid a, b \in \mathbb{Z} \}$ denote the Gaussian integers. Describe the quotient ring $\mathbb{Z}[i]/(3)$. What are its elements? What sort of algebraic structure is it?

3. Let V be a vector space and $T : V \to V$ be a linear transformation (i.e., a linear operator on V). Show that if $T^2 = T$, then V is the direct sum $V = \mathbf{Im}(T) \oplus \mathbf{Ker}(T)$ of the image and the kernel of T. Describe how to pick a nice basis for V relative to this direct sum, and give the matrix of T in that basis.

Part B

4. Let S_n denote the symmetric group on $\{1, 2, \ldots, n\}$ and view $\mathsf{S}_{n-1} \subseteq \mathsf{S}_n$ as the permutations which fix n.
   (a) Is S_{n-1} a normal subgroup? Justify.
   (b) The elements in each coset have a distinguishing feature. What is it? You should be able to identify that two permutations are in the same coset by this feature.
   (c) Let S_n act on these cosets by left multiplication: $g \cdot (h\mathsf{S}_{n-1}) = (gh)\mathsf{S}_{n-1}$. This is a familiar action of the symmetric group. Describe it clearly.

5. Construct a finite field of order 9 as a quotient $E = \mathbb{F}[x]/(p(x))$. Find a generator of the multiplicative group of E.

6. A proper ideal Q of a commutative ring R is called *primary* if whenever $ab \in Q$ and $a \notin Q$ then $b^n \in Q$ for some positive integer n.
   (a) Give an example of an ideal in $\mathbb{Z}$ that is primary but not prime.
   (b) Prove: An ideal Q of R is primary if and only if every zero divisor in R/Q is nilpotent. (An element a in a ring is nilpotent if $a^n = 0$ for some positive integer n).
   (c) Use part (b) to show that (x, y^2) is a primary ideal in $\mathbb{Q}[x, y]$.

7. Let A be an abelian normal subgroup of G and write $\bar{\mathsf{G}} = \mathsf{G}/\mathsf{A}$. Show that $\bar{\mathsf{G}}$ acts on A by conjugation: $\bar{g} \cdot a = gag^{-1}$ where $\bar{g} = g\mathsf{A}$. Under what condition on G is this action faithful? (i.e., the map $\bar{\mathsf{G}} \to \mathsf{Sym}(\mathsf{A})$ is one-to-one, where $\mathsf{Sym}(\mathsf{A})$ is the group of permutations of A). Give an explicit example illustrating that this action is not well-defined if A is not abelian.

Part C

8. Let H be a subgroup of the finite group G with $n = [\mathsf{G} : \mathsf{H}]$ and let X be the set of left cosets of H in G. Let G act by multiplication on X, i.e., $g \cdot (x\mathsf{H}) = (gx)\mathsf{H}$, and let $\phi : \mathsf{G} \to \mathsf{Sym}(X)$ be the corresponding homomorphism to the permutation group of X.

   (a) Show that $\ker(\phi)$ is the largest normal subgroup of G that is contained in H.

   (b) Show that if $n!$ is not divisible by $|\mathsf{G}|$, then H must contain a nontrivial normal subgroup of G.

   (c) Let $|\mathsf{G}| = 108$. Show that G must have a normal subgroup of order 9 or 27.

9. The Galois group of $E = \mathbb{Q}(i, \sqrt[4]{2})$ over $\mathbb{Q}$ is $\mathbf{Gal}(E/\mathbb{Q}) \cong \mathsf{D}_4$, the dihedral group of order 8.

   (a) Give a basis for E as a vector space over $\mathbb{Q}$.

   (b) The dihedral group is presented by $\mathsf{D}_4 = \langle\, \sigma, \tau \,|\, \sigma^4 = 1, \tau^2 = 1, \tau\sigma = \sigma^{-1}\tau \,\rangle$. Describe the field automorphisms that correspond to σ and τ.

   (c) Find the fixed field corresponding to the subgroup $\langle \sigma \rangle$ and the fixed field corresponding to $\langle \tau \rangle$.

   (d) Show that $\langle \tau \rangle$ is conjugate to $\langle \sigma^2\tau \rangle$ and then illustrate how to use this information to find the fixed field of $\langle \sigma^2\tau \rangle$.

10. Let $\mathsf{G} = \{g_1, g_2, \ldots, g_n\}$ be a finite group with $|\mathsf{G}| = n$. As is done with the regular representation of G, label an ordered basis of $\mathbb{C}^n$ with the elements of the group $\{e_{g_1}, e_{g_2}, \ldots, e_{g_n}\}$. Only now, act on the basis by conjugation instead of left multiplication. Thus

$$\begin{array}{rccc} \rho : & \mathsf{G} & \to & GL_n(\mathbb{C}) \\ & g & \mapsto & \rho_g \end{array} \qquad \text{where} \quad \rho_g(e_h) = e_{ghg^{-1}}.$$

   (a) Prove that this is a representation.

   (b) Discuss whether this representation is faithful (sometimes, always, never, when?).

   (c) Discuss whether this representation is irreducible.

   (d) Find a nice formula for the corresponding character $\chi(g)$