Swarthmore Honors Exam 2011

Algebra

You are expected to justify your assertions and to show your work.

Do not hurry. It will be best to work carefully.

1. *Do either (i) or (ii).*

1(i) Let G denote the general linear group $GL_2(\mathbb{F}_p)$ of invertible 2×2 matrices with entries in the field $\mathbb{F}_p$ of integers modulo a prime p. Find a Sylow p-subgroup of G, and determine the number of Sylow p-subgroups.

1(ii) Prove that every group of order 48 contains a proper normal subgroup.

2. Let V be a finite-dimensional vector space with a skew-symmetric bilinear form $\langle\, ,\, \rangle$, and let W be a two-dimensional subspace of V on which the restriction to W is nondegenerate. Explain why an orthogonal projection $\pi : V \to W$ exists, and find a formula for it, in terms of a suitable basis.

3. What facts about ideals in the integer polynomial ring $\mathbb{Z}[x]$ can one derive from the homomorphism $\mathbb{Z}[x] \xrightarrow{\varphi} \mathbb{Z}[i]$ that sends x to i?

4. *Do either (i) or (ii).*

4(i) Decide whether or not the polynomial $x^4 + 9x + 9$ generates a maximal ideal in the ring $\mathbb{Q}[x]$ of polynomials with rational coefficients.

4(ii) Prove that every ideal I of the polynomial ring $R = \mathbb{C}[x_1, ..., x_n]$ that is not the whole ring is contained in a maximal ideal of R.

5. *Do any two of the three parts.*

5(i) Let $\alpha_1, \alpha_2, \alpha_3$ denote the three complex roots of the polynomial $f(x) = x^3 - x + 1$, listed in arbitrary order. Prove that $\alpha_1^k + \alpha_2^k + \alpha_3^k$ is a rational number, for every positive integer k.

5(ii) The notation is as in (i). To compute in the splitting field $K = \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$ of $f(x)$ using the symbols α_i, one must be able to decide when two expressions in the roots are equal elements of K. Explain how this might be done.

5(iii) Let K denote the field $\mathbb{C}(t)$ of rational functions in a variable t. This field has an automorphism σ that sends t to it^{-1}. Determine the fixed field of the cyclic group $< \sigma >$ generated by this automorphism.

6. *Do either (i) or (ii).*

6(i) Gauss proved that a regular 17-gon can be constructed with ruler and compass. Explain what goes into this theorem, and prove as much as time permits.

6(ii) Let $V \xrightarrow{f} W$ be a linear transformation of real vector spaces, let $k(f)$ denote the dimension of the kernel (the nullspace) of f, and let $c(f)$ denote the dimension of the quotient space $W/\text{image}(f)$. If $k(f)$ and $c(f)$ are finite, the *index* of f is defined to be the difference $i(f) = k(f) - c(f)$. The index is not defined when $k(f)$ or $c(f)$ is infinite.

(a) Assuming that V and W are finite-dimensional, say $\dim V = m$ and $\dim W = n$, what values are possible for the index $i(f)$?

(b) Let $U \xrightarrow{g} V \xrightarrow{f} W$ be linear transformations. Prove that $i(fg) = i(f) + i(g)$, provided that the terms are defined. If you can do so, prove this without assuming that the spaces U, V, W are finite-dimensional.