Swarthmore Honors Exam 2010
Algebra
Prof Michael Artin, MIT

There is a choice on some questions. If time remains, you can try another one for extra credit, but do not hurry. It will be best to work carefully.

You are expected to justify your assertions and to show your work.

The total number of points is 110. *Within each problem, the parts will be weighted equally.*

1. (15 points) Let G be a nonabelian group of order 28 whose Sylow 2 subgroups are cyclic.

(a) Determine the numbers of Sylow 2-subgroups and of Sylow 7-subgroups.

(b) Determine the numbers of elements of G of each order.

(c) Prove that there is at most one isomorphism class of such groups.

2. (15 points) This problem is about the space V of real polynomials in two variables x, y. The fact that it is an infinite-dimensional space plays a role only in part (c).

If f is a polynomial, ∂_f will denote the operator $f(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$, and $\partial_f(g)$ will denote the result of applying this operator to a polynomial g. We also have the operator of multiplication by f, which we write as m_f. So $m_f(g) = fg$.

The rule $\langle f, g\rangle = \partial_f(g)_0$ defines a bilinear form on V, the subscript 0 denoting evaluation of a polynomial at the origin. For example, $\langle x^2, x^3\rangle = \partial_x^2(x^3)_0 = (6x)_0 = 0$.

(a) Prove that this form is symmetric and positive definite, and that the monomials $x^i y^j$ form an orthogonal basis of V (not an orthonormal basis).

(b) Linear operators A and B on V are adjoint if $\langle Ap, q\rangle = \langle p, Bq\rangle$ for all polynomials p and q. Prove that ∂_f and m_f are adjoint operators.

(c) When $f = x^2 + y^2$, the operator ∂_f is the Laplacian, which is often written as Δ. A polynomial h is harmonic if $\Delta h = 0$. Let be the space H of harmonic polynomials. Identify the space $H^\perp$ orthogonal to H with respect to the given form.

3. *(10 points) Do either (a) or (b).*

(a) Describe the maximal ideals in $\mathbb{Z}[x]$.

(b) Determine the number of irreducible polynomials of degree 4 in $\mathbb{F}_3[x]$.

4. *(10 points) Do either (a) or (b).*

(a) Let $\varphi : \mathbb{Z}[x] \to \mathbb{C}$ be the homomorphism that sends x to a complex number γ. Prove that the kernel of φ is a principal ideal.

(b) Let $f(x) = x^5 + cx^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ be an integer polynomial such that $a_i \equiv 0$ modulo 3 for all i and that $a_0 \not\equiv 0$ modulo 9. There is no condition on the coefficient c of x^4 other than that it is an integer. Prove that f is irreducible in $\mathbb{Z}[x]$ unless it has an integer root.

5. *(10 points) Do any two of the three parts.*

Let R denote the ring of Gauss integers: $R = \{a + bi \,|\, a, b \in \mathbb{Z}\}$.

(a) A Gauss prime is a Gauss integer that has no proper Gauss integer factor and is not a unit in R. Factor $3 + 9i$ into Gauss primes.

(b) Let M denote the additive group $(\mathbb{Z}/5\mathbb{Z})^+$. In how many ways can M be given the structure of an R-module?

(c) Solve $AX = B$ for X in R^2, when

$$A = \begin{pmatrix} 1+i & i \\ 2 & 1+i \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

6. (25 points) Let $\alpha = \sqrt[3]{2}, \beta = \sqrt{3}$, and $\gamma = \alpha + \beta$. Let L be the field $\mathbb{Q}(\alpha, \beta)$, and let K be the splitting field of the polynomial $(x^3 - 2)(x^2 - 3)$ over $\mathbb{Q}$.

(a) Determine the degrees $[L : \mathbb{Q}]$ and $[K : \mathbb{Q}]$.

(b) Determine all automorphisms of the field L.

(c) Determine the degree of γ over $\mathbb{Q}$.

(d) Let f be the irreducible polynomial for γ over $\mathbb{Q}$. What are the complex roots of f?

(e) Determine the Galois group of $K/\mathbb{Q}$.

7. (25 points) Do either I or II.

These problems are about group representations. If you haven't studied them before, we recommend problem I.

Definitions: If V is a real or a complex vector space, $GL(V)$ will denote the group of invertible linear operators on V. A *representation* of a group G on V is a homomorphism $R : G \to GL(V)$. The image via R of a group element g will be denoted by R_g. When $V = \mathbb{R}^n$ or $\mathbb{C}^n$, $GL(V)$ can be identified with the general linear group GL_n of invertible (real or complex) $n \times n$ matrices, and then R is also called a *matrix representation*.

Suppose given a representation R of G on V. A subspace W is *G-invariant* if $R_g W = W$ for all g. When this is so, one can obtain a representation of G on W from R by restriction. A representation R is *irreducible* if V has no proper G-invariant subspace, and is not the zero space.

The *character* of a representation R is the function $\chi : G \to \mathbb{C}$ defined by $\chi(g) = \operatorname{trace} R_g$. The character of an irreducible representation is called an *irreducible character*. One basic property of the character is that $\chi(g) = \chi(g')$ if g and g' are conjugate group elements.

I. All representations in this problem are real.

A molecule M in 'Flatland' (a two-dimensional world) consists of three like atoms a_1, a_2, a_3 forming a triangle. The molecule is vibrating, so its shape isn't constant, but the triangle is equilateral at time t_0. In given coordinates, the center of the triangle at time t_0 is at the origin, and a_1 is on the positive x-axis. The group of symmetries of M at time t_0 will be referred to as G. It is the dihedral group D_3.

We list the velocities of the individual atoms and call the resulting six-dimensional vector $\boldsymbol{v} = (v_1, v_2, v_3)^t$ the *state* of M. The group G operates on the space V of state vectors, and this operation defines a six-dimensional matrix representation S of G. For example, the rotation ρ by $2\pi/3$ about the origin permutes the atoms cyclically, and at the same time it rotates them. So the operation of ρ on state vectors, which we denote by S_ρ, sends $(v_1, v_2, v_3)^t$ to $(\rho v_3, \rho v_1, \rho v_2)^t$.

(a) The rotation ρ and the reflection r about the x-axis generate G. Determine the matrices S_ρ and S_r.

(b) Determine the subspace W of V of vectors that are fixed by S_ρ, and show that W is a G-invariant subspace.

(c) Let S' denote the representation on W obtained from S by restriction. Determine the characters of S and S', and write them as sums of irreducible characters of G.

(d) Find an explicit decomposition of W as a direct sum of irreducible G-invariant subspaces.

(e) Explain the meanings of the subspaces found in (d) in terms of motions and vibrations of the molecule.

II. This problem is about complex representations.

Let G be the dihedral group D_5, presented with generators x, y and relations $x^5 = 1$, $y^2 = 1$, $yxy^{-1} = x^{-1}$.

(a) Determine the conjugacy classes in G.

(b) Determine the dimensions of the irreducible representations of G.

Let χ be an arbitrary two-dimensional character of G.

(c) What does the relation $x^5 = 1$ tell us about $\chi(x)$?

(d) The last relation shows that x and x^{-1} are conjugate. What does this tell us about $\chi(x)$?

(e) Determine the character table of G.