Swarthmore College
Department of Mathematics and Statistics
2009 Honors Examination in Algebra
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This exam covers more material than any single student is expected to know, or will have time to solve. Feel free to skip around. The problems are of variable difficulty, so work the problems that you find easiest first. You are invited to sketch your approach to problems that you don't have time to solve completely, and to solve parts of problems while leaving earlier or later parts unsolved.

[**1**] (a) Let H be a subgroup of a finite group G. Show that the number of conjugates gHg^{-1} of H in G divides the index $[G : H]$ of H in G.

(b) Now let G act on the left on a set X, and consider the orbit $\mathcal{O}_x = \{ gx \mid g \in G \}$ of an element $x \in X$. Let $G_x = \{ g \in G \mid gx = x \}$ denote the stabilizer (isotropy subgroup) of x. Are all of the subgroups G_y for $y \in \mathcal{O}_x$ conjugate to G_x? Does this list include every subgroup conjugate to G_x? Does each distinct subgroup appear the same number of times?

(c) A recurring theme in group theory is that we need look no further than the group G itself, to understand questions like the above. Can you construct a set Y from G itself which admits a left action by G, and has an orbit with the same structure as $\mathcal{O}_x$ above?

[**2**] (a) Classify the finite groups of order 12.

(b) Two ~~such~~ groups are the nonsingular matrices of the form A with integer entries mod 2, and the nonsingular matrices of the form B with integer entries mod 3, where

$$A = \begin{bmatrix} a & b & \\ c & d & \\ & & e \end{bmatrix}, \qquad B = \begin{bmatrix} a & b \\ & c \end{bmatrix}$$

Which groups are they?

[**3**] (a) Construct explicitly the finite field $\mathbb{F}_{25}$ of order 25, as a quotient $\mathbb{F}_5[x]/(f(x))$ for a polynomial $f(x)$.

(b) In your notation, what is the irreducible polynomial over $\mathbb{F}_5$ of the element $x + 1$ of $\mathbb{F}_{25}$?

(c) Let G be the group of invertible 2×2 matrices with entries in $\mathbb{F}_{25}$. What is the order of G? How many conjugacy classes of G consist of elements of order 3? Find a representative for each of these conjugacy classes.

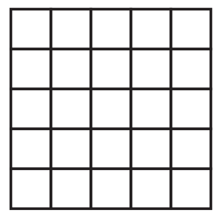
(d) Find a 3-Sylow subgroup of G. How many 3-Sylow subgroups does G have?

[4] (a) Let G be a finite group acting on a finite set X. Prove Burnside's formula: The number of orbits of this action is

$$\frac{1}{|G|}\sum_{g\in G}|X^g|$$

where X^g denotes the set of elements in X that are fixed by g.

(b) Consider an unshaded $n \times n$ checkerboard, like the board shown for $n = 5$:



We would like to count the number of ways of placing two checkers on such a board, up to symmetry, where the board may be flipped or rotated. How would you formulate this problem as a group action? What is the group G? What is the set X? How many ways are there?

(c) Identify opposite sides of the board to form a torus, which may be flipped, rotated, or translated. Now what is the group G? How many ways can we place two checkers on this torus, up to symmetry?

[5] (a) Describe the irreducible representations of the symmetric group S_3, and their character table.

(b) How many irreducible representations does the symmetric group S_4 have?

(c) How many irreducible representations does the alternating group A_5 have?

[6] (a) Let A be a square matrix that satisfies the matrix equation $A^2 - 5A + 6I = 0$, where I is the identity matrix. What is the structure of the ring $\mathbb{Z}[A]$ generated by A over $\mathbb{Z}$?

(b) Find a formula for A^n, as an element of $\mathbb{Z}[A]$.

[7] (a) Define irreducible and prime for an integral domain. What is the relationship between euclidean, unique factorization, and principal ideal domains?

(b) List a few of the smallest irreducible elements of the integral domain

$$\mathbb{Z}[\sqrt{-2}] \;=\; \{\, a + b\sqrt{-2} \;\mid\; a, b \in \mathbb{Z} \,\}$$

as measured by the norm $N(a + b\sqrt{-2}) \;=\; a^2 + 2b^2$. (In the complex numbers, this is length squared.)

(c) Is $\mathbb{Z}[\sqrt{-2}]$ a unique factorization domain?

(d) More generally, consider the integral domain

$$\mathbb{Z}[\sqrt{-c}] \;=\; \{\, a + b\sqrt{-c} \;\mid\; a, b \in \mathbb{Z} \,\}$$

where c is a positive integer. For which of these domains does the integer 6 factor uniquely into irreducibles?

[8] Let $\mathbb{C}[s,t]$ be the ring of polynomials on $\mathbb{C}^2$, and let $\mathbb{C}[w,x,y,z]$ be the ring of polynomials on $\mathbb{C}^4$. In other words, each element f of $\mathbb{C}[s,t]$ is a polynomial function $f : \mathbb{C}^2 \to \mathbb{C}$, and each element g of $\mathbb{C}[w,x,y,z]$ is a polynomial function $g : \mathbb{C}^4 \to \mathbb{C}$.

Consider the map $\alpha : \mathbb{C}^2 \to \mathbb{C}^4$ given by $\alpha(s,t) = (s^3, s^2t, st^2, t^3)$. The map α induces a ring homomorphism $\beta : \mathbb{C}[w,x,y,z] \to \mathbb{C}[s,t]$ given by the rule $g \mapsto g \circ \alpha$. In other words, we restrict polynomials on $\mathbb{C}^4$ to the image of α, and pull back by α to obtain polynomials on $\mathbb{C}^2$.

(a) Describe the image of β as a subring $R \subset \mathbb{C}[s,t]$.

(b) Describe the kernel of β as an ideal $I \subset \mathbb{C}[w,x,y,z]$.

(c) An ideal P in a commutative ring R is prime if and only if the corresponding quotient ring R/P is an integral domain. Show that I is a prime ideal in $\mathbb{C}[w,x,y,z]$. Find a strictly ascending chain $I \subset J \subset K$ of prime ideals in $\mathbb{C}[w,x,y,z]$, where K is a maximal ideal.

[9] For any cubic polynomial $f(x) = (x-\alpha)(x-\beta)(x-\gamma)$, the discriminant D of f is defined to be

$$D = (\alpha-\beta)^2(\alpha-\gamma)^2(\beta-\gamma)^2$$

(a) Derive a formula for D in terms of p and q, for polynomials of the form $f(x) = x^3 + px + q$.

(b) Prove that the discriminant of an irreducible cubic polynomial $f(x)$ in $\mathbb{Q}[x]$ is a square in $\mathbb{Q}$ if and only if the degree of its splitting field is 3.

(c) What is the degree of the splitting field of $f(x) = x^3 - 7x + 7$?

[10] (a) Prove the Eisenstein Criterion: If $f(x) \in \mathbb{Z}[x]$ and p is a prime, such that the leading coefficient of $f(x)$ is not divisible by p, every other coefficient of $f(x)$ is divisible by p, but the constant term is not divisible by p^2, then $f(x)$ is irreducible in $\mathbb{Q}[x]$.

(b) Show that $x^5 - 81x + 3$ is irreducible in $\mathbb{Q}[x]$.

(c) Can $x^5 - 81x + 3 = 0$ be solved by radicals?