

Swarthmore College
Department of Mathematics and Statistics
2008 Honors Examination in Algebra

This exam covers more material than any single student is expected to know; feel free to skip around. Take no more than an hour to do as much of #1 (True/False) as you can; then try to give detailed solutions to four of the other problems. Beyond that, feel free to give partial solutions on any of the remaining problems. I am interested in seeing how you would approach a problem, even if you don't have time to write out complete details.

1. Decide whether each of the following statements is true or false. Give brief reasons or counterexamples to support your answer (full details unnecessary). If you have time, try to salvage any false statements (if appropriate), adding conditions or reworking the statement so that it would be true. **Read the statements carefully!**
 - (a) Let G be a group. If d divides the order of G , then G has a subgroup of order d .
 - (b) In a commutative ring, the intersection of any two prime ideals is also a prime ideal.
 - (c) The polynomial $4x^2 + 6x + 3$ is a unit in $\mathbb{Z}_8[x]$.
 - (d) Any subring of a field is also a field.
 - (e) For every integer $a > 0$, $x^3 + ax + 1$ is irreducible in $\mathbb{Q}[x]$.
 - (f) The number of \mathbb{Q} -automorphisms of $\mathbb{Q}(\sqrt[4]{2})$ is 4.
 - (g) The permutations $(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{smallmatrix})$ and $(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 4 & 3 & 2 \end{smallmatrix})$ are conjugate in the symmetric group S_5 .
 - (h) The factorizations $10 = 2 \cdot 5$ and $10 = (1 + 3i)(1 - 3i)$ show that $\mathbb{Z}[i]$ is not a unique factorization domain.
 - (i) Every unique factorization domain is a principal ideal domain.
 - (j) A regular 340-gon is constructible using only straightedge and compass.
 - (k) Every symmetric polynomial in $\mathbb{C}[x_1, \dots, x_n]$ can be written uniquely as a linear combination of elementary symmetric polynomials.
 - (l) The matrix $\begin{pmatrix} i & i \\ i & 1 \end{pmatrix}$ is normal.

2. A subgroup H of a group G is called a *characteristic subgroup* if $\phi(H) = H$ for all automorphisms ϕ of G .

(a) Prove that every subgroup of a cyclic group is characteristic.

(b) Prove that the center of a group is characteristic.

(c) If H is a characteristic subgroup of K and K is a characteristic subgroup of G , must H be a characteristic subgroup of G ?

3. For any commutative ring R , let $\text{Aff}(R) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a, b \in R, a \neq 0 \right\}$.

(a) For an odd integer $m > 1$, show that the commutator subgroup of $\text{Aff}(\mathbb{Z}/(m))$ is $\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle$.

(b) Compute the number of p -Sylow subgroups of the group $\text{Aff}(\mathbb{Z}/(7))$ for $p = 3, 5, 7$.

4. Construct a field of size 9, and find a generator for its nonzero elements.

5. Let G be a group of order pq , where $p < q$ are primes with $q \not\equiv 1 \pmod{p}$. Show that G must be cyclic.

6. Let $A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ define a bilinear form on \mathbb{R}^3 by $\langle v, w \rangle = v^t A w$.

(a) Find a nonzero vector $v \in \mathbb{R}^3$ which is self-orthogonal, i.e., $\langle v, v \rangle = 0$.

(b) Find an orthogonal (not necessarily orthonormal) basis of \mathbb{R}^3 for this bilinear form and compute the matrix for the bilinear form with respect to that basis.

7. Let $X = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$ be the set of all 2-element subsets of $\{1, 2, 3, 4\}$. Let each permutation $\sigma \in S_4$ act on the set X by the rule

$$\sigma(\{i, j\}) = \{\sigma(i), \sigma(j)\},$$

where $i \neq j$.

(a) Decompose X into orbits for the action of the subgroup of S_4 generated by (12).

(b) Decompose X into orbits for the action of the subgroup of S_4 generated by (123).

(c) Prove all the elements of S_4 act on X by even permutations.

8. Let $I = (2, x)$ be the ideal generated by 2 and x in $\mathbb{Z}[x]$.
- (a) Show that I is not principal.
- (b) Show that I is maximal. (This does not need part (a).)
9. Let $\zeta = e^{2\pi i/8}$ be a primitive 8th root of unity.
- (a) Find the degree of the extension $\mathbb{Q}(i, \sqrt[4]{2})$ over \mathbb{Q} and a \mathbb{Q} -basis for it.
- (b) Is $\mathbb{Q}(i, \sqrt[4]{2}) = \mathbb{Q}(i + \sqrt[4]{2})$? Explain.
- (c) Find the degree of the extension $\mathbb{Q}(\zeta, \sqrt[4]{2})$ over \mathbb{Q} .
- (d) Is $\mathbb{Q}(\zeta, \sqrt[4]{2}) = \mathbb{Q}(\zeta + \sqrt[4]{2})$? Explain.
10. An incomplete character table of a finite group G is given below; the number in parentheses above each element indicates the size of that element's conjugacy class. For example, the conjugacy classes of G are represented by 1, u , v , w , x , and y , and the conjugacy class of v contains 2 elements.

| $\#C_g$ | (1) | (1) | (2) | (2) | (3) | (?) |
|----------|-----|-----|-----|-----|------|-----|
| g | 1 | u | v | w | x | y |
| χ_1 | 1 | 1 | 1 | 1 | 1 | 1 |
| χ_2 | 1 | 1 | 1 | 1 | -1 | ? |
| χ_3 | 1 | -1 | 1 | -1 | i | ? |
| χ_4 | 1 | -1 | 1 | -1 | $-i$ | ? |
| χ_5 | 2 | 2 | -1 | -1 | 0 | ? |
| χ_6 | ? | ? | ? | ? | ? | ? |

- (a) Complete the character table and find the size of G .
- (b) Show that u has order 2 and x has order 4.
- (c) Show that v generates a normal subgroup of order 3.
- (d) Show the representation corresponding to χ_6 is faithful.
- (e) Show that w has order 6.