Swarthmore College

Department of Mathematics and Statistics

2007 Honors Examination in Algebra

Instructions: This exam contains 10 problems. Try to solve about *six* problems as completely as possible. Beyond that, turn in any solutions or partial solutions that you can get done. I am interested in your thoughts on the problem even if they do not completely solve it. In particular, turn in your solution even if you can't do all the parts of a multiple part problem. You might also formulate and solve special cases that you can think of. Where there are multiple parts to a problem, you can answer a later part without solving all the earlier ones. The problems are not necessarily in increasing order of difficulty.

1. Let S_n denote the symmetric group of permutations of $\{1, 2, \ldots, n\}$. Consider $S_3 \subseteq S_4$ in the natural way such that the permutations in S_3 fix the number 4.
   (a) Find a set of coset representatives of S_3 in S_4 (i.e., one from each coset).
   (b) Find a property on the permutations of S_4 such that σ and τ share this property if and only if they are in the same coset.
   (c) Generalize both (a) and (b) to S_n (with proof).

2. Prove from basic principles that in a commutative ring R with unity 1, an ideal M is maximal if and only if R/M is a field.

3. Show that $x^3 - x - 1$ and $x^3 - x + 1$ are irreducible over $\mathbb{Z}_3$ (the integers mod 3). Construct their splitting fields and explicitly exhibit an isomorphism between these splitting fields.

4. Let G be a group with subgroup H. For $g \in G$ define the (H, H)-double coset of g to be

$$HgH = \{\ h_1 g h_2 \mid h_1, h_2 \in H\ \}.$$

   (a) Show that the (H, H) double cosets partition G but that, unlike left cosets, different (H, H)-double cosets need not have the same cardinality.
   (b) Suppose that G acts transitively on the set $X = \{x_1, \ldots, x_n\}$ and that $H = \{g \in G \mid gx_1 = x_1\}$. Show that the number of orbits of X under the action of H is the same as the number of (H, H)-double cosets. (Hint: since G acts transitively, there exist elements $g_i \in G$ such that $g_i x_1 = x_i$).

5. True or False? Justify your answers.
   (a) The principal ideal $\langle 1 + \sqrt{-5} \rangle$ is maximal in $\mathbb{Z}[\sqrt{-5}]$.
   (b) $\mathbb{Z}[i]/\langle 2 - i \rangle \cong \mathbb{Z}_5$, (Here, $\mathbb{Z}[i]$ denotes the ring of Gaussian integers).
   (c) The ring $S \subseteq \mathbb{Z}[x]$ consisting of polynomials with zero coefficient on the linear term

$$S = \left\{a_0 + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n \mid a_i \in \mathbb{Z}\right\}$$

   is a unique factorization domain (hint: think about x^2 and x^3).

6. Let E be the splitting field of the polynomial $x^3 - 2$ over $\mathbb{Q}$.
   (a) Find a basis for E as a vector space over $\mathbb{Q}$.
   (b) Find a primitive element α such that $E = \mathbb{Q}(\alpha)$. (Note that it suffices to show that α is not contained in any proper subfield of E.)

7. (a) Let G be a group and $H \leq G$ a subgroup with $[G : H] = n$. Prove that there exists a normal subgroup $N \triangleleft G$ such that

$$\text{(a)} \quad [G : N] \text{ divides } n! \qquad \text{and} \qquad \text{(b)} \quad N \subseteq H$$

   (Hint: Let G act on the set of left cosets $X = \{gH | g \in G\}$ to get $\phi : G \to S_n$.)

   (b) Use the previous result to show that any group of order 24 must have a normal subgroup of order 4 or 8. (Hint: let H be a Sylow-2 subgroup.)

8. Let R be a commutative ring with unity 1. An element t in an R-module M is called a torsion element if $rt = 0$ for some nonzero element $r \in R$. The set of torsion elements in M is denoted $tor(M)$.
   (a) Prove that if R is an integral domain then $tor(M)$ is a submodule of M and show that $M/tor(M)$ has no nonzero torsion elements (i.e., it is torsion free).
   (b) Give an example of a ring R and an R-module M such that $tor(M)$ is not a submodule.
   (c) Show that if R has zero divisors, then every nonzero R-module has torsion elements.

9. Every permutation in S_n is either even or odd. Define the sign of a permutation σ to be $\text{sign}(\sigma) = 1$, if σ is even, and $\text{sign}(\sigma) = -1$, if σ is odd. Prove that

$$\text{sign}(\sigma) = \prod_{1 \leq i < j \leq n} \frac{\sigma(i) - \sigma(j)}{i - j}.$$

10. Let G be a finite group with conjugacy classes $C_1, C_2, \ldots, C_\ell$. Let $\mathbb{C}G = \{\sum_{g \in G} \lambda_g g | \lambda_g \in \mathbb{C}\}$ denote the group algebra of G over the complex numbers $\mathbb{C}$.
   (a) Show that the class sums $\overline{C_i} = \sum_{c \in C_i} c$ are in the center $Z(\mathbb{C}G)$.
   (b) Let V be a G-module over $\mathbb{C}$. For $z \in Z(\mathbb{C}G)$, show that the map $\phi_z : V \to V$ given by $\phi_z(v) = zv$ is a G-module homomorphism.
   (c) Suppose that V is an irreducible G-module. What does Schur's lemma say about the homomorphism ϕ_z? Use this to describe multiplication of vectors in V by an element z of the center.
   (d) As an example, let V be the permutation module of S_4 spanned by v_1, v_2, v_3, v_4 with $\sigma(v_i) = v_{\sigma(i)}$. Let $W = \text{span}(v_2 - v_1, v_3 - v_1, v_4 - v_1)$. Then W is an irreducible submodule of V (you do not need to prove this). Let C be the conjugacy class consisting of two-cycles. Describe the action of $\overline{C}$ on W. If possible, generalize to S_n.