Swarthmore College
Department of Mathematics and Statistics
Honors Examination: Algebra
Spring 2005

Instructions: This exam contains twelve problems. Try to solve six problems as completely as possible. Do not be concerned if some problems are unfamiliar; you have a lot of choice so that one exam can cover several syllabi. Once you are satisfied with your answers to your six problems, make a second pass through the exam and complete as many parts of the remaining problems as possible. I am interested in your thoughts on a problem even if you do not completely solve it. In particular, turn in your solution even if you can't do all the parts of a multiple part problem. You might also formulate and solve special cases if you cannot solve a problem in the most general case. When there are multiple parts to a problem, you can answer a later part without solving all the earlier ones.

1. Which of the following groups are finitely-generated? If the group is finitely-generated, find a minimal set of generators. Prove your assertions.

(a) $\mathbb{Z}_2 \times \mathbb{Z}_2$
(b) $\mathbb{Z}_2 \times \mathbb{Z}_3$
(c) $\mathbb{Z}_2 \times \mathbb{Z}$
(d) $\mathbb{Q}^\times$, the multiplicative group of nonzero rational numbers
(e) S_4
(f) A_4

2. Let K be an arbitrary field.

(a) Show that a polynomial $f(x) \in K[x]$ of degree three is irreducible if and only if it has no roots in K.
(b) Show that $3x^5 + 4x^3 + 14x + 2$ is irreducible in $\mathbb{Q}[x]$.
(c) Show that the $x^4 + 4x^2 + 9$ is reducible in $\mathbb{Z}_2[x]$ and $\mathbb{Z}_3[x]$ (in fact, this polynomial is reducible over $\mathbb{Z}_p$ for all primes p, though you are not being asked to prove this fact).
(d) Show that $x^4 + 4x^2 + 9$ is irreducible in $\mathbb{Q}[x]$.

3. Let G be a finite group. Given a subset $S \subset G$, the conjugates of S are the sets of the form $g^{-1}Sg$, where $g \in G$. Given a subgroup H of G, define the normalizer of H by

$$N(H) = \{g \in G \mid g^{-1}Hg = H\}.$$

(a) Prove that $N(H)$ is a subgroup of G.
(b) Prove that H is a normal subgroup of $N(H)$.
(c) Show that G is not equal to the union of the conjugates of H, for any proper subgroup H.
(d) Give an example of a proper subset S of a finite group such that the group is the union of the conjugates of S.

4. Let $F = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, where none of d_1, d_2, and $d_1 d_2$ are perfect squares in $\mathbb{Q}$.

(a) Prove that $F/\mathbb{Q}$ is a normal extension.
(b) Prove that the Galois group $\mathrm{Gal}(F/\mathbb{Q})$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$.
(c) Let $L/\mathbb{Q}$ be a normal extension with Galois group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Prove that L is of the form $L = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, where none of d_1, d_2, and $d_1 d_2$ are squares.

5. Let K be a field of characteristic $p \neq 0$. Denote the algebraic closure of K by $\overline{K}$ and define

$$K^{1/p^\infty} = \{\alpha \in \overline{K} \mid \alpha^{p^\ell} \in K \text{ for some } \ell \geq 0\}.$$

(a) Prove that K^{1/p^∞} is a subfield of $\overline{K}$.
(b) Show that p^ℓth roots are unique in $\overline{K}$; that is, if $\alpha^{p^\ell} = \beta^{p^\ell}$, then $\alpha = \beta$.
(c) Let F be an algebraic extension of K. Show that if there is a homomorphism of F into K^{1/p^∞} that fixes K, then there is exactly one homomorphism of F into $\overline{K}$ that fixes K.
(d) Prove the converse of (c).

6. Let R be a ring. Denote the set of all infinite sequences of elements of R by $R[[x]]$. Given two sequences $\mathbf{c} = (c_0, c_1, c_2, \dots)$ and $\mathbf{d} = (d_0, d_1, d_2, \dots)$, we define addition by

$$\mathbf{c} + \mathbf{d} = (c_0 + d_0, c_1 + d_1, c_2 + d_2, \dots),$$

and we define multiplication by setting the nth component of $\mathbf{c} \cdot \mathbf{d}$ to be

$$\sum_{i=0}^{n} c_i d_{n-i},$$

where $n = 0, 1, 2, \dots$.
(a) Prove that $R[[x]]$ is a ring.
(b) Show that if $\mathbf{c} = (c_0, c_1, c_2, \dots)$ is a unit of $R[[x]]$, then c_0 is a unit in R.
(c) Show that if c_0 is a unit of R, then $\mathbf{c} = (c_0, c_1, c_2, \dots)$ is a unit of $R[[x]]$.

7. Let k be a field and $f(x) \in k[x]$. We say α is a root of $f(x)$ of multiplicity m if

$$(x - \alpha)^m \mid f(x).$$

If $f(x) = \sum_{i=0}^{d} c_i x^i$, then we define the derivative of f by $f'(x) = \sum_{i=1}^{d} c_i i x^{i-1}$.
(a) Suppose α is a root of $f(x)$. Prove that the multiplicity of α is greater than one if and only if $f'(\alpha) = 0$.
(b) Show that $x^{p^n} - x$ is the product of all monic, irreducible polynomials in $\mathbb{F}_p[x]$ whose degrees divide n.

8. True or false? Justify your answers.
(a) There is no simple group of order 18.
(b) Let p and q are distinct primes such that $p \not\equiv 1 \pmod{q}$ and $q \not\equiv 1 \pmod{p}$. Suppose G is a group of order pq. The group G is necessarily cyclic.
(c) If G is a finite abelian group of order n, then for every divisor d of n, G has a subgroup of order d.
(d) If G is a finite abelian group of order n, then for every divisor d of n, G has an element of order d.

9. If G is an abelian group, define $\widetilde{G}$ to be the collection of all homomorphisms of G into the multiplicative group of nonzero complex numbers. Given $\varphi_1, \varphi_2 \in \widetilde{G}$, define $(\varphi_1 \cdot \varphi_2)(g) = \varphi_1(g)\varphi_2(g)$.
(a) Show that $\widetilde{G}$ is an abelian group.
(b) Show that if G is a finite abelian group, then G and $\widetilde{G}$ are isomorphic.

10. Let R be a ring and let $I = \langle 2x+2 \rangle$ be an ideal of the polynomial ring $R[x, y]$, where $\langle f(x, y) \rangle$ represents the ideal generated by the polynomial $f(x, y)$.

(a) Show that if R is a UFD, then $I = \langle 2x+2 \rangle$ is a prime ideal of $R[x, y]$.
(b) Find a maximal ideal of $\mathbb{C}[x, y]$ that contains I. Justify your answer.
(c) Find a maximal ideal of $\mathbb{Z}[x, y]$ that contains I. Justify your answer.
(d) Does $\mathbb{C}[x, y]$ contain a maximal ideal of the form $\langle f(x, y) \rangle$? Why or why not?

11. Let $\rho : G \to GL(n, \mathbb{C})$ be a representation of G. For $g \in G$, let $\delta(g)$ be the determinant of $\rho(g)$.

(a) Show that δ is a character of degree one.
(b) Suppose ρ is the regular representation of G and $g \in G$. Prove that if g has even order r and $|G|/r$ is odd, then $\delta(g) = -1$; otherwise, $\delta(g) = 1$.
(c) Show that G has a normal subgroup of index two if $-1 \in \operatorname{im}(\delta)$.
(d) Suppose $|G| = 2m$, where m is odd. Show that G has a normal subgroup of index two.

12. Determine the validity of each of the following statements. As usual, you must completely justify your claims.

(a) The ring of integers of $\mathbb{Q}[\sqrt{-3}]$ is $\mathbb{Z}[\sqrt{-3}]$.
(b) The ring of integers of $\mathbb{Q}[\sqrt{3}]$ is $\mathbb{Z}[\sqrt{3}]$.
(c) The ring of integers of $\mathbb{Q}[\sqrt{7}]$ is $\mathbb{Z}[\sqrt{7}]$.
(d) The ring of integers of $\mathbb{Q}[\sqrt{3}, \sqrt{7}]$ is $\mathbb{Z}[\sqrt{3}, \sqrt{7}]$.