Swarthmore College

Department of Mathematics and Statistics

Honors Examination

Algebra

May 2001

Instructions: Do about six of the following problems. Turn in your solutions even if you have not done all the parts of the problem.

1. Let N be a normal subgroup of the group G. Prove the following:

(a) If N has a trivial center and G/N has a trivial center, then G has a trivial center.

(b) Let p be a prime. If the order of every element of N is a power of p, and the order of every element of G/N is a power of p, then the order of every element of G is a power of p.

2. Let S_{10} be the symmetric group of permutations of the set $\{1, 2, \ldots, 10\}$. The following problems are not necessarily related.

(a) Find an element of maximum order in S_{10}. Justify your answer.

(b) Let $\phi : S_{10} \to \mathbb{C}^*$ be a homomorphism from S_{10} into the multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ of complex numbers. Show that the image of ϕ is contained in $\{1, -1\}$.

3. Let $G = GL(3, \mathbb{Z}_2)$. That is, G is the set of all 3×3 invertible matrices with entries from $\mathbb{Z}_2$.

(a) Show that $|G| = 168$. Hint: for G to be invertible, the rows of G must be linearly independent and $168 = 7 \cdot 6 \cdot 4$.

(b) The conjugacy class of g in G is $cl(g) = \{x^{-1}gx \mid x \in G\}$. If N is a normal subgroup of G, show that N is the union of conjugacy classes of G (this relationship is true for any group).

(c) Consider the following elements of G.

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Show that a and b are in different conjugacy classes. One way to do it is to consider the characteristic equation of these matrices. They are $1 - x^3$ for a and $1 - x + x^2 - x^3$ for b.

(d) There are also matrices c, d, e in G such that the size of the conjugacy classes of these matrices are

$$|cl(a)| = 24, \quad |cl(b)| = 24, \quad |cl(c)| = 21, \quad |cl(d)| = 42, \quad |cl(e)| = 56.$$

Prove that G has no nontrivial, proper normal subgroups. That is, G is a simple group.

4. For a group G let $Z \doteq Z(G)$ be its center.
   (a) Show that there is a one-to-one correspondence between the subgroups of G/Z and the subgroups of G that contain Z.
   (b) Show that if G/Z is cyclic, then G is abelian (and thus $G = Z$).
   (c) The group of quaternions Q_8 can be defined by

$$Q_8 = \left\{ \pm 1, \pm i, \pm j \pm k \;\middle|\; \begin{array}{l} i^2 = j^2 = k^2 = -1, 1^2 = (-1)^2 = 1, \\ ij = -ji = k, jk = -kj - i, ki = -ik = j, \\ -i = (-1)i, -j = (-1)j, -k = (-1)k \end{array} \right\}$$

   Show that Q_8 is the union of 3 abelian subgroups I, J, K such that $I \cap J \cap K = Z(Q_8)$.
   (d) Show that $G/Z \cong Q_8$ is impossible.

5. Let I be an ideal of a commutative ring R. Define

$$\sqrt{I} = \{ \, a \in R \mid a^n \in I, \text{ for some } n \in \mathbb{Z}^+ \, \}.$$

The set $\sqrt{I}$ is called the radical of I. The special case of $\sqrt{\{0\}}$ (where $I = \{0\}$ is the zero ideal) is called the nilradical.
   (a) Show that $\sqrt{I}$ is an ideal of R.
   (b) Show by examples that it is possible to have $\sqrt{I} = I$ and possible to have $\sqrt{I} \neq I$.
   (c) Find (with proof) the nilradical of $R/\sqrt{I}$.
   (d) Find (with proof) the relation between $\sqrt{I}/I$ and the nilradical of R/I.

6. True or False? Justify your answers. The isomorphisms below are ring isomorphisms, and $\langle f \rangle$ denotes the principal ideal generated by f.

   (a) $\dfrac{\mathbb{Z}_7[x]}{\langle x^2+3 \rangle}$ is a field. (b) $\dfrac{\mathbb{Q}[x]}{\langle x^2+1 \rangle} \cong \dfrac{\mathbb{Q}[x]}{\langle x^2-1 \rangle}$. (c) $\dfrac{\mathbb{Z}_2[x]}{\langle x^3+x+1 \rangle} \cong \dfrac{\mathbb{Z}_2[x]}{\langle x^3+x^2+1 \rangle}$.

7. Let R be a commutative ring with unity. Let M denote the set of non-units of R (that is, the set of noninvertible elements of R). Suppose that M is an ideal of R.
   (a) Prove that R/M is a field.
   (b) Prove that M is a unique maximal ideal of R.
   (c) Prove that if $r \in R$, then either r or $1 - r$ is invertible.
   (d) Prove that if p is a prime, then the set of non-units of $R = \mathbb{Z}/p^2\mathbb{Z}$ is an ideal.

8. Let R be a commutative ring with unity that has the property that every ideal I of R is prime.
   (a) Prove that R is an integral domain.
   (b) Prove that R is a field. Hint: choose a clever ideal.
   (c) Is the converse of (b) true? The converse is: if R is a field, then every ideal of R is prime.