Swarthmore College

Department of Mathematics and Statistics

Honors Examination

7 May 1998

Algebra

(1) How many elements of order 6 are there in the symmetric group S_6?

(2) How many 2×2 matrices A with entries in $\mathbb{Z}/7$ are there with $A^5 = I$?

(3) Are there any homomorphisms of S_4 onto S_3? Of S_5 onto S_4?

(4) Let p be a prime with $p \equiv 3 \pmod 4$. If p is divided into $\left(\frac{p-1}{2}\right)!$, what can the remainder possibly be?

(5) Let $\mathbb{Z}_{1000}^{\times}$ be the group of units in the ring $\mathbb{Z}_{1000}$. What is the largest order any element of $\mathbb{Z}_{1000}^{\times}$ can have?

(6) In $\mathbb{Z}[i]/(79)$, what is $(9+i)^{80}$? (Hint: show that $(a+bi)^{79} = a^{79} + (bi)^{79}$.)

(7) In $(\mathbb{Z}/17)[x]$ factor $x^{16} - 1$, $x^8 - 1$, and $x^8 + 1$ into irreducible factors.

(8) (a) Show that $(\mathbb{Z}/2)[x]/(x^4 + x^3 + 1)$ is a field.

(b) Find the remainder when x^{200} is divided by $x^4 + x^3 + 1$ in $(\mathbb{Z}/2)[x]$.

(9) $6 = 2 \cdot 3 = \sqrt{6} \cdot \sqrt{6}$. Nevertheless, $\mathbb{Z}[\sqrt{6}]$ has unique factorization. Resolve the apparent contradiction.

(10) Write down a ring homomorphism from $\mathbb{Z}[\sqrt{10}]$ onto $\mathbb{Z}/2$. Is the kernel of your homomorphism a principal ideal?

(11) Is 1 a linear combination $x^2 - 2$ and $x^3 - 3$ in $\mathbb{Z}[x]$? Is 1 a linear combination of $x^2 - 1$ and $x^3 - 2$ in $\mathbb{Z}[x]$?