SENIOR HONORS EXAM IN ABSTRACT ALGEBRA

Swarthmore College

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INSTRUCTIONS:
Please do 6 of the following problems as thoroughly as you can. Choose at least one problem from each of the following four sections:
(1) group theory, (2) ring theory, (3) field theory/Galois theory, and (4) linear algebra. If you have time, please do some of the remaining problems.

SECTION I: GROUP THEORY

(1). Let A(4) denote the alternating group on 4 letters.

(a). Find all normal subgroups of A(4).

(b). Find a 2-Sylow subgroup of A(4).

(c). How many 2-Sylow subgroups are contained in A(4) ?

(d). Find the largest abelian quotient group of A(4).

(2). Let p be an odd prime.

(a). Show that every group of order p^2 is abelian.

(b). Show that there exists a group of order 2p which is not abelian.

(3). (a). Give the definition of "solvable group".

(b). Prove that any group whose order is a prime power is solvable.

(c). Let p and q be distinct prime numbers. Show that any group of order $(p^2)(q)$ is solvable (Hint: Use part (b) above.)

(d). Give an example of a non-solvable group. Please justify your answer.

SECTION II: RING THEORY

(4). (a). Show that the group of units in the ring $Z/2^r Z$ is cyclic if r = 1, or 2.
(b). Show that the group of units in the ring $Z/2^r Z$ is not cyclic if r > 2.
(c). Is the ring of units in Z/pZ cyclic for p prime? Please justify your answer.

(5). Suppose that R is a commutative ring with 1 and that R contains an element e such that e is neither 1 nor 0, and $e^2 = e$. (Thus e is an idempotent.) Show that R contains at least two distinct maximal ideals.

(6). If R is an integral domain which has finitely many elements, show that R is a field.

SECTION III: GALOIS THEORY

(7). Let q(x) be the polynomial $x^7 - 1$. A root of this polynomial is given by $z = \cos(2\pi/7) + i\sin(2\pi/7)$.
(a). Consider the Galois group of Q(z + 1/z) over Q. Show that this Galois group is isomorphic to Z/3Z.
(b). Exhibit a polynomial over Q such that the Galois group of its' splitting field is Z/3Z. (Use part (a) above.)
(c). List some other finite groups which are Galois groups of splitting fields for polynomials over Q. Please give some explanation/justification for your answer. Complete proofs are not necessary.

(8). Consider the polynomial $p(x) = x^4 - 4x^2 - 1$ in Q[x].
(a).Find the splitting field in $\mathbb{C}$ of p(x).
(b). Determine the Galois group G for the splitting field of the polynomial p(x) over Q.
(c). Give a proper non-identity subgroup H of G and describe the intermediate field associated to your choice of subgroup H. (Thus H is neither G nor {1}.)

(9). Exhibit a specific polynomial of degree 5 over the rational numbers which is not solvable by radicals. Explain why it is not solvable by radicals.

SECTION IV: LINEAR ALGEBRA

(10). Let A be a linear transformation from R^n to R^n. Show that if A has n distinct eigenvalues (characteristic roots), then there is a basis e(1), ..., e(n) for R such that

$$A(e(i)) = x_i e(i) \text{ for some choice of real numbers } x_i .$$

(11). Prove that if A and B are n x n matrices for which A is invertible , then the product matrices AB and BA have the same set of eigenvalues.

(12). Let A be the 2x2 symmetric matrix over the real numbers given by

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

(a). Show that A has 2 real eigenvalues (characteristic roots).

(b). Exhibit a 2x2 invertible matrix B such that BAB^t is a diagonal matrix (where B^t denotes the transpose of B).

(c). Assume that A has 2 positive eigenvalues, what can you say about the sign of

$$ax^2 + 2bxy + cy^2$$

for all non-zero real numbers x and y? Please give reasons for your answer.