Swarthmore College
Department of Mathematics and Statistics
Honours Examination: Algebra
Spring 2025

Instructions This exam contains 9 problems. You are not expected to solve all of them.

I am interested in seeing how you approach the problems, so please turn in your solution to a problem even if it is incomplete.

I would rather see substantial progress on fewer problems than a bit of progress on all of them. That said, some parts of a question may be easier, or significantly harder, than others. If you get stuck, feel free to set aside the harder bits of a question, and perhaps return to it later. When you attempt a problem, I suggest reading through all its parts before you start working on it. You may solve the problems, and the parts of the problems, in any order you like.

Good luck!

1. (a) Let $\mathrm{GL}_3(\mathbb{R})$ denote the group of 3×3 invertible matrices over the real numbers (under matrix multiplication). Let G be the subgroup of $\mathrm{GL}_3(\mathbb{R})$ containing *monomial matrices* with entries ± 1. That is, a matrix $M \in \mathrm{GL}_3(\mathbb{R})$ is in G if and only if each row and column of M has one non-zero entry, and each non-zero entry of M is 1 or -1. Show that M is a subgroup of $\mathrm{GL}_3(\mathbb{R})$.

(b) Let H be a group and let N be a subgroup of H. Show that N is a normal subgroup of H if and only if there is a group K and homomorphism $\varphi : H \to K$ such that $\ker \varphi = N$.

(c) Show that the group G from part (a) has a subgroup of 24 elements. Show that this subgroup is normal in G.

2. In this problem, we think about groups of order 99.

(a) Classify all abelian groups of order 99.

(b) Prove that a group of order 99 is not simple.

(c) Classify all groups of order 99.

3. For a group G and a prime p consider the p-power torsion subset:

$$T_p(G) = \{g \in G \mid \exists n \in \mathbb{N} : \ g^{p^n} = 1_G\}.$$

(a) Show that if G is abelian then $T_p(G)$ is a subgroup of G.

(b) Give an example of a group G such that $T_p(G)$ is not a subgroup of G.

(c) Can you give an example of a group G such that for every positive integer n and every prime p the group $T_p(G)$ has an element of order p^n?

4. Consider the action of A_4 on S_4 by conjugation. For $g \in A_4$ and $x \in S_4$ we have $g.x = gxg^{-1}$.

(a) Describe the orbits of this action.

(b) Describe the conjugacy classes inside the group A_4.

(c) How many (pairwise non-isomorphic) irreducible representations does A_4 have?

(d) What are the dimensions of the irreducible representations of A_4?

5. For each of the following rings R, answer the following three questions. How many ideals does this ring have? How many of these are prime ideals? How many of them are maximal ideals?

(a) $R = \mathbb{Z}/2025\mathbb{Z}$

(b) $R = \mathbb{Z}/45\mathbb{Z} \times \mathbb{Z}/45\mathbb{Z}$

(c) $R = \mathbb{Z}/81\mathbb{Z} \times \mathbb{Z}/25\mathbb{Z}$

6. In this question, we work with commutative, unital rings R. By an *irreducible element* of the ring, we mean an element $r \in R$ such that if $r = ab$ for $a, b \in R$ then a or b is a unit in R. By a *prime element* of the ring, we mean an element $p \in R$ such that if $a, b \in R$ and $p \mid ab$ then $p \mid a$ or $r \mid b$.

(a) Give an example of a (commutative, unital) ring S and an element $c \in S$ such that c is irreducible in S but the ideal cS is not maximal.

(b) Define what it means for a (commutative, unital) ring to be a principal ideal domain.

(c) Show that if R is a principal ideal domain and $c \in R$ then the following are equivalent:

(i) c is an irreducible element

(ii) c is a prime element

(iii) cR is a maximal ideal

(iv) cR is a prime ideal

7. Let F be a field of 49 elements.

(a) How many polynomials of the form $x^2 + a \in F[x]$ are irreducible?

(b) How many polynomials of the form $x^2 + bx + c \in F[x]$ are irreducible?

(c) How many polynomials of the form $x^3 + d \in F[x]$ are irreducible?

(d) Prove that $x^3 + d \in F[x]$ is irreducible if and only if $d^{16} \neq 1$. (Here 1 denotes the multiplicative identity in the field F.)

8. Let $\alpha = \sqrt{-3 + \sqrt{3}}$.

(a) Find the minimal polynomial f for α over $\mathbb{Q}$.

(b) Let K be the splitting field of f (from part (a)) over $\mathbb{Q}$. What is the Galois group $\mathrm{Gal}(K|\mathbb{Q})$? (Identify it as an abstract group.)

(c) Describe the intermediate fields of the extension $K|\mathbb{Q}$.

9. Let $K|F$ be a Galois extension with Galois group $\mathrm{Gal}(K|F)$ and assume $\alpha \in K$ is a primitive element, i.e. $K = F(\alpha)$. Let the minimal polynomial of α over F be

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in F[x].$$

Consider the following three quantities.

(a) $N_a = (-1)^n a_0$

(b) $N_b = \prod_{\sigma \in \mathrm{Gal}(K|F)} \sigma(a)$

(c) $N_c = \det(m_a)$, where $m_a : K \to K$ is the F-linear map $m_a(k) = ak$.

Explain the relationship of these quantities. (Compare all three, if you are able to, and justify your answer.)