Swarthmore College
Department of Mathematics and Statistics
Geometry Exam 2025

Work on at least one problem from each section and complete as many questions as you can. There is no expectation that you complete all the problems.

1 Curves

1. Consider the curve $\alpha(t) = (\sin 2t, t, \cos 2t)$.

   (a) Determine the curvature and torsion of the curve α.

   (b) For α, find the equation of the osculating plane (i.e the plane determined by the normal and the tangent vectors) when $t = \pi/4$.

   (c) Find the Frenet trihedron (tangent, normal and binormal vectors) for the curve in part (a), when $t = \pi/2$.

2. Let $\alpha(s)$ be a curve parametrized by arclength with nowhere vanishing curvature κ and torsion τ.

   (a) Suppose $\alpha(s)$ lies on the surface of a sphere centered at the origin. Prove that
   $$R^2 + (R^{'})^2 T^2$$
   is constant, where $R = \frac{1}{\kappa}, T = \frac{1}{\tau}$, and $R^{'}$ is the derivative with respect to s.

   (b) Prove that the converse of part (a) is also true.

3. Let $\alpha : [0,1] \to \mathbb{R}^3$ be a parametrized curve such that $\alpha^{'}(t) \neq 0, \forall t \in [0,1]$. Prove that $|\alpha(t)|$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha^{'}(t), \forall t \in [0,1]$.

2 Surfaces

4. (a) Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be a differentiable function, let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation, and let $p \in \mathbb{R}^2$. Prove that
   $$d(T \circ f)|_p = (T \circ df)|_p,$$
   where d denotes the differential and $\circ$ denotes composition.

   (b) Let S be a regular surface and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be an invertible linear transformation. Use part (a) to prove that $T(S)$ is a regular surface.

(c) Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be a function given by

$$f(u, v) = (u \sin v, u \cos v, 2v).$$

Compute the differential df and prove that $f(\mathbb{R}^2)$ is a regular surface.

5. Prove that all the tangent planes of a surface given by $z = xf(y/x), x \neq 0$, where f is a differentiable function, all pass through the origin $(0, 0, 0)$.

3 Curvature on Surfaces

6. Let C be a regular curve on a surface S with Gaussian curvature $K > 0$. Show that the curvature κ of C at p satisfies

$$\kappa \geq \min\{k_1, k_2\},$$

where k_1 and k_2 are the principal curvatures of S at p.

7. Compute:

   (a) First fundamental form **I**

   (b) Second fundamental form **II**

   (c) The Gauss map

   for the hyperboloid $z = xy$ at $(1, 1, 1)$.

4 Manifolds

8. Let $G = \{M = \begin{bmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \in Mat(3 \times 3, \mathbb{R}) | \det(M) \neq 0\}$.

   (a) Prove that G is a differentiable manifold.

   (b) What is the dimension of G? Justify your answer with a proof.

   (c) Give an example of a 2-dimensional emdedded submanifold of G.

9. Let $\widetilde{M}$ and M be smooth connected differentiable manifolds. A map $\pi : \widetilde{M} \to M$ is a covering map if π is smooth and surjective, and each point in M has a neighbourhood U such that π maps each component of $\pi^{-1}(U)$ diffeomorphically onto U. Suppose that M has a Riemannian metric g. Show that $\widetilde{M}$ has a Riemannian metric $\widetilde{g}$ such that the covering map $\pi : \widetilde{M} \to M$ is a local isometry.