Part I — Real Analysis

I-1. Give explicit examples of each of the following (no explanation needed), or explain why no example is possible.

(a) A nonempty bounded subset of $\mathbb{R}$ that does not contain its supremum.

(b) An infinite subset of $\mathbb{R}$ consisting entirely of isolated points. (If $S \subseteq \mathbb{R}$, a point x of S is *isolated* if there is an open interval $I \subseteq \mathbb{R}$ centered at x with the feature that $I \cap S = \{x\}$).

(c) A bounded and infinite subset of $\mathbb{R}$ consisting entirely of isolated points.

(d) A closed, bounded, and infinite subset of $\mathbb{R}$ consisting entirely of isolated points.

(e) A subset of $\mathbb{R}$ that is both open and closed.

(f) A subset of $\mathbb{R}$ that is neither open nor closed.

I-2. Suppose that X is a metric space with metric d, and that $A \subseteq X$. Recall that p is called a *limit point* or a *cluster point* of A if, given any $\varepsilon > 0$, there is some point $a \in A$ for which

$$a \neq p \text{ but } d(p, a) < \varepsilon.$$

(An equivalent formulation of this definition is the following: p is a limit point of A if there is a sequence (a_n) of points in $A \setminus \{p\}$ that converges to p.)

Note carefully that p does not have to belong to A to be a limit point of A.

Let $L(A)$ be the set of limit points of A:

$$L(A) = \{p \in X \ : \ p \text{ is a limit point of } A\}.$$

Prove that $L(A)$ is a closed subset of X.

I-3. Suppose that X is a metric space with metric d.

(a) Choose and fix some point $p \in X$. Show that the function $f_p : X \to \mathbb{R}$ given by the formula

$$f_p(x) = d(p, x)$$

is continuous.

(b) Choose and fix a compact set $K \subseteq X$. Given any $x \in X$, let us define the "distance from x to K" as

$$D_K(x) = \min_{y \in K} d(x, y).$$

Show that the function $D_K : X \to \mathbb{R}$ is well-defined and continuous.

Remark: Part (a) tells us that, in some sense, "distance is continuous"; this is not surprising. It turns out that, if we use d to define a metric on the Cartesian product $X \times X$ in the standard way, then the metric d is in fact a continuous function from $X \times X$ to $\mathbb{R}$.

I-4. Let us define the function $f : [1, \infty) \to [1, \infty)$ by

$$f(x) = \frac{x+2}{x+1}.$$

(a) Show that, for any $x > 1$,

$$|f'(x)| \leq \frac{1}{4}.$$

(b) Use part (a) to conclude that, given any two $x, y \in [1, \infty)$,

$$|f(x) - f(y)| \leq \frac{|x-y|}{4}.$$

(c) Use part (b) to conclude that any sequence (a_n) of the form

$$a_0 = \text{ some number in } [1, \infty), \quad a_n = f(a_{n-1}) \text{ for all } n \in \mathbb{N}$$

converges to $\sqrt{2}$.

I-5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, and that $[a, b] \subseteq \mathbb{R}$ is an interval with $a < b$. Choose and fix $\delta > 0$, define the function $g_\delta : (a, b) \to \mathbb{R}$ by the formula

$$g_\delta(x) = \frac{1}{\delta} \int_x^{x+\delta} f(t)\, dt.$$

(a) Show that g_δ is differentiable on (a, b).

(b) Show that, given any $\varepsilon > 0$, δ can be chosen small enough that

$$|f(x) - g_\delta(x)| < \varepsilon \text{ for all } x \in (a, b).$$

Remark: The point here is that a continuous function can be approximated arbitrarily closely on a bounded interval by a differentiable function.

Part II — Real Analyis II

II-1. Prove that any linear transformation is its own derivative. More specifically, prove the following: if T is an $m \times n$ matrix, viewed as a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, and $\mathbf{a}$ is any point in $\mathbb{R}^n$, then

$$DT(\mathbf{a}) = T.$$

II-2. Let $A \subseteq \mathbb{R}^2$ be an open set. Suppose that $f : A \to \mathbb{R}^2$ is a C^1 function with the feature that

$$|D_1 f_1(\mathbf{x})| < \frac{1}{2},\ |D_2 f_1(\mathbf{x})| < \frac{1}{2},\ |D_1 f_2(\mathbf{x})| < \frac{1}{2},\ |D_2 f_2(\mathbf{x})| < \frac{1}{2}$$

for all $\mathbf{x} \in A$. (Here $D_i f_j(\mathbf{x})$ denotes the ith partial derivative of the jth component function of f at the point $\mathbf{x}$.)

Now consider the map $F : A \to \mathbb{R}^2$ given by the formula $F(\mathbf{x}) = \mathbf{x} + f(\mathbf{x})$. (Intuitively, the idea here is that F is a "perturbation" of the identity map.)

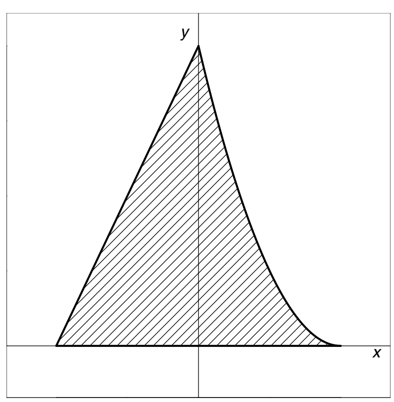
(a) Show that F is locally one-to-one — that is, given any $\mathbf{x} \in A$, there is some open set $U \subseteq A$ containing $\mathbf{x}$ such that F is one-to-one on U.

(b) Now assume further that A is *convex* — that is, given any two points in A, the line segment joining them lies entirely in A. Show that F is one-to-one on all of A.

II-3. Let S be the open square $(0,2) \times (0,2)$ in $\mathbb{R}^2$, and consider the function $T : S \to \mathbb{R}^2$ given by

$$T(x,y) = \big((x-y), xy^2\big).$$

$T(S)$ is pictured below.



(a) Assuming that T is one-to-one on S (which it is), find the area of $T(S)$.

(b) Prove that T is one-to-one on S.

II-4. A smooth function $u : \mathbb{R}^2 \to \mathbb{R}$ is called *harmonic* if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

everywhere in $\mathbb{R}^2$.

Suppose that $u : \mathbb{R}^2 \to \mathbb{R}$ is harmonic, let C be a circle of radius r about the point (x_0, y_0), and let ω be the 1-form

$$\omega(x,y) = \frac{\partial u(x,y)}{\partial x}\, dy - \frac{\partial u(x,y)}{\partial y}\, dx.$$

(a) Compute $d\omega$ (show the steps as explicitly as you can).

(b) Use either Green's or Stokes' theorem, together with the fact that u is harmonic, to show that

$$\int_C \omega = 0.$$

(c) Consider the following parameterization of C:

$$p(t) : [0, 2\pi] \to \mathbb{R}^2, \quad \text{where} \quad p(t) = (x_0 + r\cos(t), y_0 + r\sin(t)).$$

Use this parameterization to write

$$\int_C \omega$$

explicitly as an integral in t from 0 to 2π.

(d) Use the same parameterization as in part (c) to write

$$\frac{1}{2\pi r} \int_C u\, dV.$$

explicitly as an integral in t from 0 to 2π.

[Here we are using Munkres' notation: this is the integral of u over C "with respect to one-dimensional volume" or "with respect to arc length", divided by the length of the circle. We interpret this integral as the *average value of u on C*.]

(e) Drawing on your work in parts (b) and (c) and (d), show that

$$\frac{\partial}{\partial r}\left(\frac{1}{2\pi r}\int_C u\, dV\right) = 0.$$

[Hint: since everything in sight is smooth, you may and should move the $\partial/\partial r$ under the integral sign.]

(f) Use part (e) and the observation that

$$\lim_{r\to 0^+} \frac{1}{2\pi r}\int_C u\; dV = u(x_0, y_0)$$

to explain (in just a sentence or two) the following conclusion: *the average value of u on any circle is equal to the value of u at the center of the circle.*

Remark: This compelling conclusion is sometimes called the *circle-mean-value property for harmonic functions*, and provides some insight into why harmonic functions are used as models of physical quantities that are in some sense "fully diffused."

II-5. Let $U \subset \mathbb{R}^{n-1}$ be an open set, and $\alpha : U \to \mathbb{R}^n$ a smooth coordinate patch (in particular, α has rank $n-1$ everywhere). Let $W \subset \mathbb{R}^n$ be an open set such that

$$\alpha(U) = W \cap g^{-1}(0)$$

for some smooth function $g : \mathbb{R}^n \to \mathbb{R}$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function, and suppose that the restriction of f to $\alpha(U)$ has a strict maximum: that is, there is some $p \in U$ such that

$$f(\alpha(p)) > f(\alpha(q)) \text{ for all } q \in U \setminus p.$$

(a) Prove that $Df(\alpha(p)) = \lambda Dg(\alpha(p))$ for some real number λ. (Here $Df(\alpha(p))$ is the Jacobian of f at $\alpha(p)$, *not* the Jacobian of $f \circ \alpha$ at p! Similarly for $Dg(\alpha(p))$.)

(b) Use part (a) to find the maximum of the function $f(x, y, z) = xyz$ on the set $\alpha(U)$, where

$$U = \{(x, y) \;:\; 0 < x < 1 \text{ and } 0 < y < 1\ \}$$

and

$$\alpha(x, y) = \left(x, y, 1 - \frac{x}{2} - \frac{y^2}{2}\right).$$

[Hint number 1: Take $g(x, y, z) = z - (1 - x/2 - y^2/2)$. Hint number 2: f certainly won't be maximized where either x or y is zero, so you may assume that x and y are nonzero.]

Remark: This is essentially the method of *Lagrange multipliers*.