Topology Honors Exam 2022
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Instructions: Do as many of these problems as you can. You should aim to do a balance from the two sections (problems 1–6 and problems 7–12). If you think that the solution to a problem is a one-liner, "There is a theorem that says this," then don't just cite it — provide a proof of that theorem. (Problem 4(a) is an example.)

Section 1

1. Let $\mathbb{Z}_+$ be the set of positive integers and let $\mathcal{P}(\mathbb{Z}_+)$ be the set of all subsets of $\mathbb{Z}_+$. Given $y \in \mathcal{P}(\mathbb{Z}_+)$ and $n \in \mathbb{Z}_+$, let

$$U(y,n) = \{x \in \mathcal{P}(\mathbb{Z}_+) \,:\, x \cap \{1,2,\ldots,n\} = y \cap \{1,2,\ldots,n\}\}.$$

   Prove that the sets $U(y,n)$ are a basis for a Hausdorff topology on $\mathcal{P}(\mathbb{Z}_+)$.

2. Let X be a topological space, and define an equivalence relation $\sim$ on X by setting $x_1 \sim x_2$ if $f(x_1) = f(x_2)$ for every continuous function f from X to a Hausdorff space. Let $HX = X/\sim$ with the quotient topology.
   (a) For spaces A and B, let $\operatorname{Map}(A,B)$ denote the set of continuous functions $A \to B$. Prove that for any space X and any Hausdorff space Y, there is a bijection

$$\operatorname{Map}(X,Y) \leftrightarrow \operatorname{Map}(HX,Y).$$

   (b) Prove that HX is Hausdorff.

3. (a) Prove that if X is second-countable, then X has a countable dense subset.
   (b) Prove that if X is metrizable and has a countable dense subset, then X is second-countable.

4. Let X be a compact metric space.
   (a) Prove that every sequence in X has a convergent subsequence.
   (b) Prove the Lebesgue Number Lemma: if $\mathcal{A}$ is an open cover of X, there exists $\delta > 0$ such that every subset of X of diameter less than δ is contained in a member of $\mathcal{A}$. (The *diameter* of a set S is $\sup\{d(x,y) \,:\, x, y \in S\}$.)

5. Prove (just using the basic definitions, and in particular without using Tychonoff's theorem) that the product of two compact spaces is compact.

6. Let $I = [0,1]$ be the unit interval, let X be a topological space, and consider maps $I \to X$.
   (a) Prove that if X is path connected, then any two maps $I \to X$ are homotopic. (I mean homotopic, not path homotopic.)
   (b) Give an example to show that conclusion in (a) fails if X is connected but not path connected.

Section 2

7. Let X be the infinite earring space:

$$X = \bigcup_{n \geq 1} C_n$$

where C_n is the circle of radius $1/n$ centered at $(1/n, 0)$. Let Y be the wedge of countably many circles: Y is a space which is a union of subspaces S_n, $n = 1, 2, 3, \ldots$, each of which is homeomorphic to the unit circle, and there is a point $y \in Y$ such that $S_i \cap S_j = \{y\}$ whenever $i \neq j$. Endow Y with the *coherent* topology: a subset C of Y is open if and only if $C \cap S_n$ is open for each n.

   Are X and Y homeomorphic? Are they homotopy equivalent? Explain.

8. (a) A function $X \to Z$ is said to *factor through* Y if f can be written as a composite $X \to Y \to Z$. Prove that any map that factors through a contractible space is null-homotopic.
   (b) Prove that any map $f : \mathbb{R}P^2 \to T$ is null-homotopic. (Hint: consider the universal cover $\mathbb{R}^2 \to T$.)
   (c) Prove that any map $f : S^n \to S^m$ is null-homotopic if $n < m$. If you can't do this in general, do the case when $n = 1$.

9. Let S be a surface (compact, without boundary) such that every simple closed curve on S separates S. What are the possibilities for S?

10. If A is a subspace of a topological space X, a *retraction* of X onto A is a map $r : X \to A$ such that the composite $A \xrightarrow{i} X \xrightarrow{r} A$ is the identity on A, where $i : A \to X$ is the inclusion map.

    It is a fact that every compact surface S without boundary can be realized as a subspace of $\mathbb{R}^4$. (That is, there is a subspace of $\mathbb{R}^4$ which is homeomorphic to the surface.) Assume that S is not homeomorphic to S^2. Prove that there can not be a retraction $\mathbb{R}^4 \to S$, no matter how S is included as a subspace in $\mathbb{R}^4$.

    Can you say anything about the case when S is homeomorphic to S^2?

11. Let X be the one point union of a torus and a 2-sphere.
    (a) Compute $\pi_1(X)$.
    (b) Describe the universal cover of X.

12. Suppose that X is a topological space. Without using the Seifert-van Kampen Theorem, prove that if A and B are simply connected open subsets of X and $A \cap B$ is path connected, then $A \cup B$ is simply connected.