Algebraic Geometry Exam

Spring 2022

This exam consists of 10 problems. You are not required to solve them all, but rather should work on the ones you find interesting and approachable.

I am interested in seeing how you approach the various problems, so please turn in your solutions to a problem even if you can only make progress on some of the individual parts. If you find it useful, you may assume an earlier part of a problem when working on the later parts even if you haven't been able to solve it. However, I would rather see substantial progress on a few problems than a handful of computations for every problem.

For all problems you may assume the ground field is $k = \mathbb{C}$.

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1. Consider the affine plane curves $Y = V(y - x^2) \subset \mathbb{A}^2$ and $Z = V(xy - 1) \subset \mathbb{A}^2$.
   (a) Show that Y is not isomorphic to Z.
   (b) If f is an irreducible quadratic polynomial in $k[x, y]$, one can show that $V(f)$ is either isomorphic to Y or to Z (the proof is a little tedious and involves a number of cases). For $f = xy + 3x + 2y + 5$, is $V(f)$ isomorphic to Y or Z?

2. Let $Y = V(x^2 - yz, xz - x)$ be an algebraic set in $\mathbb{A}^3$. Show that Y is a union of three irreducible components; describe each component and find their prime ideals.

3. Consider the curve Y, given by $x^3 = y^2 + x^4 + y^4$ in $\mathbb{A}^2$.
   (a) Determine all the singular points, multiplicities and tangent lines of Y.
   (b) Show that the curve $\tilde{Y}$ obtained by blowing up Y at $(0, 0)$ is nonsingular.

4. (a) Fix a degree $d > 0$. How many monomials in $x_0, x_1, \ldots, x_n$ of degree d exist? Call this number N and list the monomials $m_0, \ldots, m_{N-1}$.
   (b) Show that the d-uple embedding of P^n, defined by
   $$\begin{aligned} \nu_d : \mathbb{P}^n &\to \mathbb{P}^{N-1} \\ (x_0 : x_1 : \cdots : x_n) &\mapsto (m_0 : m_1 : \cdots : m_{N-1}) \end{aligned}$$
   is a well-defined function.
   (c) Let Y be the *Veronese surface*, defined to be the image of the 2-uple embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$. Show that Y is an algebraic set.

5. Show that any rational map $\phi : \mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ is everywhere defined, hence a morphism.

6. Consider the quadric surface $Q = V(xy - zw)$ in $\mathbb{P}^3$.
   (a) Show that Q is birational to $\mathbb{P}^2$.
   (b) The Segre map $\Sigma_{1,1} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ is defined by $((s:t),(u:v)) \mapsto (su : tv : sv : tu)$. Verify that the Segre map $\Sigma_{1,1}$ is injective and the image is Q.
   (c) Show that Q is not isomorphic to $\mathbb{P}^2$.

7. Given a variety X, let $\mathrm{Sing}(X)$ denote the set of singular points of X.
   (a) Show that if $C = V(f)$ and $D = V(g)$ are curves in $\mathbb{P}^2$, then
   $$\mathrm{Sing}(C \cup D) = \mathrm{Sing}(C) \cup \mathrm{Sing}(D) \cup (C \cap D).$$
   (b) Use (a) to show that every nonsingular projective plane curve is irreducible.
   (c) Is the same statement true for affine curves? That is, is every nonsingular curve in the affine plane irreducible?

8. Consider the elliptic curve E over $\mathbb{C}$ defined by $y^2 = x^3 + x^2 - 2x$; denote by $+$ the group law on E, and O the inflection point $(0 : 1 : 0)$. Let $P = (-\frac{1}{2}, -\frac{3\sqrt{2}}{4})$ and $Q = (0, 0)$.
   (a) Verify that P and Q are on E and compute $P + Q$.
   (b) Find a point R on E such that $P + R = O$.

9. Let $Y = V(I) \subseteq \mathbb{P}^n$ be an algebraic set of dimension r with graded projective coordinate ring $S(Y) = k[x_0, x_1, \ldots, x_n]/I = \oplus_d S_d$, where S_d is the degree d part of S. Recall that the Hilbert polynomial $H_Y \in \mathbb{Q}[x]$ of Y is the polynomial with the property that $H_Y(d) = \dim S_d$ for d sufficiently large. The *degree* of Y is defined as $r!$ times the leading coefficient of H_Y, and the *arithmetic genus* of Y (denoted $p_a(Y)$) is defined to be $p_a(Y) := (-1)^r(H_Y(0) - 1)$.
   (a) Compute the Hilbert polynomial, degree and arithmetic genus of $\mathbb{P}^n$.
   (b) Let Y be a hypersurface of degree d in $\mathbb{P}^n$, compute the Hilbert polynomial and arithmetic genus of Y.

10. A variety Y of dimension r in $\mathbb{P}^n$ is called a *strict complete intersection* if the ideal $I(Y)$ can be generated by $n - r$ homogenous elements. Y is called a *set-theoretic complete intersection* if Y can be written as the intersection of $n - r$ hypersurfaces.
   (a) Show that a strict complete intersection is a set-theoretic complete intersection.
   (b) Let $T \subset \mathbb{P}^3$ be the "twisted cubic curve," which is given by the parametrization
   $$\mathbb{P}^1 \to \mathbb{P}^3,\ (s:t) \mapsto (s^3 : s^2t : st^2 : t^3)$$
   Show that T can be written as the set-theoretic intersection of the quadric $Q = V(y^2 - xz)$, and the cubic $C = V(z^3 + xw^2 - 2yzw)$. In particular, T is a set-theoretic complete intersection.

(c) Show that T is the intersection of the three quadrics:

$$Q_1 = V(xz - y^2), Q_2 = V(xw - yz), Q_3 = V(yw - z^2).$$

Show that the intersection of any two of these quadrics will not give the twisted cubic. In particular, $I(T)$ contains three elements of degree 2.

(d) Possibly using (c), explain that the ideal of the twisted cubic $I(T)$ cannot be generated by two elements, hence T is not a strict complete intersection.