Part I — Real Analysis

I-1. Give explicit examples of each of the following (no explanation needed), or explain why no example is possible.

(a) A nonempty bounded subset of $\mathbb{R}$ that does not contain its supremum.

(b) An infinite subset of $\mathbb{R}$ consisting entirely of isolated points. (If $S \subseteq \mathbb{R}$, a point x of S is *isolated* if there is an open interval $I \subseteq \mathbb{R}$ centered at x with the feature that $I \cap S = \{x\}$).

(c) A bounded and infinite subset of $\mathbb{R}$ consisting entirely of isolated points.

(d) A closed, bounded, and infinite subset of $\mathbb{R}$ consisting entirely of isolated points.

(e) A subset of $\mathbb{R}$ that is both open and closed.

(f) A subset of $\mathbb{R}$ that is neither open nor closed.

I-2. Suppose that X is a metric space with metric d, and that $A \subseteq X$. Recall that p is called a *limit point* or a *cluster point* of A if, given any $\varepsilon > 0$, there is some point $a \in A$ for which

$$a \neq p \text{ but } d(p,a) < \varepsilon.$$

(An equivalent formulation of this definition is the following: p is a limit point of A if there is a sequence (a_n) of points in $A \setminus \{p\}$ that converges to p.)

Note carefully that p does not have to belong to A to be a limit point of A.

Let $L(A)$ be the set of limit points of A:

$$L(A) = \{p \in X \ : \ p \text{ is a limit point of } A\}.$$

Prove that $L(A)$ is a closed subset of X.

I-3. Suppose that X is a metric space with metric d.

(a) Choose and fix some point $p \in X$. Show that the function $f_p : X \to \mathbb{R}$ given by the formula

$$f_p(x) = d(p,x)$$

is continuous.

(b) Choose and fix a compact set $K \subseteq X$. Given any $x \in X$, let us define the "distance from x to K" as

$$D_K(x) = \min_{y \in K} d(x,y).$$

Show that the function $D_K : X \to \mathbb{R}$ is well-defined and continuous.

Remark: Part (a) tells us that, in some sense, "distance is continuous"; this is not surprising. It turns out that, if we use d to define a metric on the Cartesian product $X \times X$ in the standard way, then the metric d is in fact a continuous function from $X \times X$ to $\mathbb{R}$.

I-4. Let us define the function $f : [1, \infty) \to [1, \infty)$ by

$$f(x) = \frac{x+2}{x+1}.$$

(a) Show that, for any $x > 1$,

$$|f'(x)| \leq \frac{1}{4}.$$

(b) Use part (a) to conclude that, given any two $x, y \in [1, \infty)$,

$$|f(x) - f(y)| \leq \frac{|x-y|}{4}.$$

(c) Use part (b) to conclude that any sequence (a_n) of the form

$$a_0 = \text{ some number in } [1, \infty), \quad a_n = f(a_{n-1}) \text{ for all } n \in \mathbb{N}$$

converges to $\sqrt{2}$.

I-5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, and that $[a, b] \subseteq \mathbb{R}$ is an interval with $a < b$. Choose and fix $\delta > 0$, define the function $g_\delta : (a, b) \to \mathbb{R}$ by the formula

$$g_\delta(x) = \frac{1}{\delta} \int_x^{x+\delta} f(t)\ dt.$$

(a) Show that g_δ is differentiable on (a, b).

(b) Show that, given any $\varepsilon > 0$, δ can be chosen small enough that

$$|f(x) - g_\delta(x)| < \varepsilon \text{ for all } x \in (a, b).$$

Remark: The point here is that a continuous function can be approximated arbitrarily closely on a bounded interval by a differentiable function.

Part II — Complex Analysis

II-1. (a) Show that the function $f(z) = z$ is analytic everywhere. If possible, do this in at least two different ways.

(b) Show that the function $g(z) = \overline{z}$ is not analytic on any open set. If possible, do this in at least two different ways.

II-2. Find the power series for $f(z) = \dfrac{1}{z^2}$ centered at $z_0 = -1$. Then use this power series to calculate the sum of the series

$$\sum_{n=0}^{\infty}(n+1)\Big(-\frac{1}{2}\Big)^n$$

(be sure to explain, as part of your proof, how you know the series converges).

II-3. Show that, if C is any circle in the complex plane that does not pass through 0, then

$$\int_C \frac{ze^z - e^z}{z^2}\, dz = 0.$$

II-4. Let us write $D(0,1)$ for the open unit disk, $C(0,1)$ for the unit circle, and

$$\overline{D(0,1)} = D(0,1) \cup C(0,1).$$

Suppose that $U \subset \mathbb{C}$ is a domain containing $\overline{D(0,1)}$ and that $f : U \to \mathbb{C}$ is analytic. Suppose that f maps $\overline{D(0,1)}$ strictly into itself: that is,

$$|f(z)| < 1 \quad \text{for all } |z| \le 1.$$

Prove that f has a *fixed point* on $D(0,1)$ — that is, there is some $p \in D(0,1)$ with the feature that $f(p) = p$.

Remark: This is a very special case of the *Brouwer fixed point theorem*, which says that any continuous self-mapping of a closed ball in $\mathbb{R}^n$ has a fixed point.

II-5. Write $D(0,1)$ for the open unit disk, and suppose that $(f_k : D \to \mathbb{C})$ is a sequence of analytic functions converging uniformly on $D(0,1)$ to a function $f : D \to \mathbb{C}$. It is a fact that, in this situation, f is analytic.

Prove that the sequence (f_k') converges normally to f' on $D(0,1)$. (Recall that saying that (f_k') converges *normally* to f' on $D(0,1)$ means that (f_k') converges to f uniformly on all closed disks contained in $D(0,1)$).

Remark: The analogous result in the real case does not hold: even if a sequence of differentiable functions $(f_k : I \to \mathbb{R})$ converges uniformly to a differentiable $f : I \to \mathbb{R}$, where I is some interval, the sequence (f_k') need not converge.